C. State-Space Realization of Transfer Function Matrix

Definition 3.3.1

A state-space model (A, B, C, D) is a realization of a TF matrix $G(s) \in \mathcal{R}^{p\times m}$ if

$$G(s) = C(sI - A)^{-1}B + D \equiv \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

The realization is called a minimal realization if *n=dim(A)* is the smallest possible dimension of all the realizations of *G(s)*.

Note:

The realization of G(s) is not unique. For example, any similarity transformation of (A, B, C, D) to another realization $(\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D})$ is also a realization of G(s).

Similarity transformation:

$$\tilde{A} = T^{-1}AT, \tilde{B} = T^{-1}B, \tilde{C} = CT, \tilde{D} = D$$

in which T is the invertible matrix that relates the two set of coordinates.

Theorem 3.3.1

(a) A state-space realization (A, B, C, D) of G(s) is minimal iff (A, B) is controllable and (C, A) is observable.

(b) *The dimension of the minimal realization is the same as the McMillan degree of G(s).*

(c) *For minimal realizations,* $P_p(s) = |sI - A|.$
For other realizations, the poles of G(s) are only a subset of the eigenvalues of A, i.e., $P_p(s)$ *is a factor of* $|sI - A|.$ #

Consider a single-input-multi-output (SIMO) system described by

$$G(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d \quad \in \mathcal{R}^{p\times 1}$$

where $a_i \in R,\ \beta_i \in R^p,\ d \in R^p$. A realization of G(s) is then given by

$$G(s) = \left[\begin{array}{c|c} A & b \\ \hline C & d \end{array}\right], \quad A \in R^{n\times n}, \quad b \in R^n$$

which is normally called **controllable canonical** form

where

Controllable Canonical Form:

$$A = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad \mathrm{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{n-1} & \beta_n \end{bmatrix} \in R^{p \times n}$$

Dually, consider a multi-input-single-output (MISO) system by

$$G(s) = \frac{\eta_1^T s^{n-1} + \eta_2^T s^{n-2} + \cdots + \eta_{n-1}^T s + \eta_n^T}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} + d^T \in \mathcal{R}^{1 \times m}$$

where $a_i \in R$, $\eta_i \in R^m$, $d \in R^m$. An **observable canonical** form realization of G(s) is then given by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|c} -a_1 & 1 & 0 & & \cdots 0 & \eta_1^T \\ -a_2 & 0 & 1 & & \cdots 0 & \eta_2^T \\ \vdots & \vdots & \vdots & & & \\ -a_{n-1} & 0 & 0 & \cdots & 1 & \eta_{n-1}^T \\ -a_n & 0 & 0 & \cdots & 0 & \eta_n^T \\ \hline 1 & 0 & 0 & \cdots & 0 & d^T \end{array}\right]$$

For a MIMO system $G(s) \in \mathcal{R}^{p \times m}$, the simplest and most straight-forward way is as follows, in which a separate realization is found for each column (or each row) $g_i(s) \in \mathcal{R}^{p \times 1}$ of G(s) using the SIMO controllable canonical form for each input u_i (or the SOMI observable canonical form for each output) and these separate realizations are then assembled. Namely, let $g_i(s)$ be the *i-th* column of G(s):

$$G(s) = \left[g_1(s) \;\; g_2(s) \;\; \cdots \;\; g_m(s) \right]$$

and put $g_i(s)$ in the form of

$$g_i(s) = \frac{1}{d_i(s)} n_i(s) + \delta_i$$

where $d_i(s) \in \mathcal{P}$ is the monic common-denominator of $g_i(s)$ given by:

$$d_i(s) = s^{k_i} + d_1^i \, s^{k_i-1} + \cdots + d_{k_i-1}^i \, s + d_{k_i}^i$$

$n_i(s) \in \mathcal{P}^{p\times 1}$ is a vector of polynomials, each of degree less than k_i given by

$$n_i(s) = \beta_1^i \, s^{k_i-1} + \beta_2^i \, s^{k_i-2} + \cdots + \beta_{k_i-1}^i \, s + \beta_{k_i}^i \, , \qquad \beta_j^i \in R^{p\times 1}$$

and $\delta_i \in R^{p\times 1}$ is a constant vector. Then, the controllable canonical form realization of the SIMO system $Y(s) = g_i(s)\, u_i(s)$ is:

$$g_i(s) = \left[\begin{array}{c|c} A_i & b_i \\ \hline C_i & d_i \end{array}\right], \qquad A_i \in R^{k_i \times k_i} \quad b_i \in R^{k_i}, \quad C_i \in R^{p \times k_i} \quad d_i \in R^p$$

$$A_i = \begin{bmatrix} -d_1^i & -d_2^i & \cdots & -d_{k_i-1}^i & -d_{k_i}^i \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad b_i = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$C_i = \begin{bmatrix} \beta_1^i & \beta_2^i & \cdots & \beta_{k_{i-1}}^i & \beta_{k_i}^i \end{bmatrix}, \quad d_i = \delta_i$$

A realization of G(s), i.e., the MIMO system

$$Y(s) = G(s)\,U(s) = \sum_{i=1}^{m} g_i(s)\,u_i(s)$$

is thus given by (A, B, C, D) where

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|cccc} A_1 & & & & b_1 & & & \\ & A_2 & & & & b_2 & & \\ & & \ddots & & & & \ddots & \\ & & & A_m & & & & b_m \\ \hline C_1 & C_2 & \cdots & C_m & D_1 & D_2 & \cdots & D_m \end{array}\right]$$

***Note*:**

The above realization is usually not minimal. To obtain a minimal realization, Kalman canonical decomposition should be used to obtain the controllable and observable subspace of the above realization for a reduced order model.

***MATLAB*:**

MINREAL Minimal realization and pole-zero cancellation.

MSYS = MINREAL(SYS) produces, for a given LTI model SYS, an equivalent model MSYS where all cancelling pole/zero pairs or non minimal state dynamics are eliminated. For state-space models, MINREAL produces a minimal realization MSYS of SYS where all uncontrollable or unobservable modes have been removed.

MSYS = MINREAL(SYS,TOL) further specifies the tolerance TOL used for pole-zero cancellation or state dynamics elimination. The default value is TOL=SQRT(EPS) and increasing this tolerance forces additional cancellations.

For a state-space model SYS=SS(A,B,C,D),

[MSYS,U] = MINREAL(SYS)

also returns an orthogonal matrix U such that (U*A*U',U*B,C*U') is a Kalman decomposition of (A,B,C).

Definition 3.3.2

A complex number $z_o \in C$ is called an invariant zero of a system realization if it makes the system matrix defined by

$$Q(s) \equiv \begin{bmatrix} A - sI_n & B \\ C & D \end{bmatrix}$$

to lose rank, i.e.,

$$\text{rank} \begin{bmatrix} A - z_o I_n & B \\ C & D \end{bmatrix} < \text{normal rank} \begin{bmatrix} A - sI_n & B \\ C & D \end{bmatrix}$$

Facts:

(a) The invariant zeros are not changed by constant state-feedback.

For any constant state feedback

$$u = Fx + v$$

the resulting system is

$$\begin{bmatrix} \dot{x} \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

$$= \begin{bmatrix} A+BF & B \\ C+DF & D \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix}$$

Noting that $\forall s \in C$,

$$\text{rank}\begin{bmatrix} A+BF-sI & B \\ C+DF & D \end{bmatrix} = \text{rank}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}\begin{bmatrix} I_n & 0 \\ F & I_m \end{bmatrix}$$

$$= \text{rank}\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$$

Thus the invariant zeros of the original system (A, B, C, D) are exactly the same as those of the new system by state feedback

(b) The invariant zeros are not changed by similarity transformation.

Lemma 3.3.1

Assume $\begin{bmatrix} A-sI & B \\ C & D \end{bmatrix}$ has full-column normal rank. Then $z_o \in C$ is an invariant zero of a realization iff $\exists\, x_o \in C^n,\ x_o \neq 0,$ and $u_o \in C^m$ such that

$$\begin{bmatrix} A-z_oI & B \\ C & D \end{bmatrix} \begin{bmatrix} x_o \\ u_o \end{bmatrix} = 0$$

Moreover, if $u_o = 0,$ then, x_o is also an unobservable mode.

Proof:

$\Rightarrow$

Suppose z_o is the invariant zero. Then

$$\begin{bmatrix} A - z_o I & B \\ C & D \end{bmatrix}$$

will not be full-column rank. Thus,

$$\exists \begin{bmatrix} x_o \\ u_o \end{bmatrix} \neq 0 \quad \text{s.t.} \quad \begin{bmatrix} A - z_o I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_o \\ u_o \end{bmatrix} = 0$$

If $x_o = 0$, then the above is equivalent to

$$\begin{bmatrix} B \\ D \end{bmatrix} u_o = 0$$

As $\begin{bmatrix} B \\ D \end{bmatrix}$ is of full-column rank by assumption, $u_o = 0$ which leads to a contradiction. Thus $x_o \neq 0$.

Moreover, when $u_o = 0$, the condition becomes:

$$\begin{bmatrix} A - z_o I \\ C \end{bmatrix} x_o = 0, \;\; x_o \neq 0 \quad \Leftrightarrow \quad \begin{cases} A x_o = z_o x_o, \;\; x_o \neq 0 \\ C x_o = 0 \end{cases}$$

which indicates that $\lambda = z_o$ is an unobservable mode with an eigenvector x_o.

$\Leftarrow$

Obvious by noting $\begin{bmatrix} x_o \\ u_o \end{bmatrix} \neq 0$ and $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ is of full column normal rank by assumption. #

***Note*:**

For square system (i.e., p=m), the invariant zero can be computed by solving a generalized eigenvalue problem given by

$$\underbrace{\begin{bmatrix} A & B \\ C & D \end{bmatrix}}_{G}\begin{bmatrix} x_o \\ u_o \end{bmatrix} = z_o \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{M}\begin{bmatrix} x_o \\ u_o \end{bmatrix} \Leftrightarrow \det\left(z_o M - G\right) = 0$$

using MATLAB command "**eig(G, M)** "

Lemma 3.3.2

Assume $\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full-row normal rank.

Then $z_o \in C$ is an invariant zero of a realization iff $\exists\, x_o \in C^n,\ x_o \neq 0,$ and $y_o \in C^p$ such that

$$\begin{bmatrix} x_o^* & y_o^* \end{bmatrix} \begin{bmatrix} A - z_o I & B \\ C & D \end{bmatrix} = 0$$

Moreover, if $y_o = 0$, then, x_o is also an uncontrollable mode. #

Fact: For any $s \notin \lambda(A)$,

$$\operatorname{rank}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \operatorname{rank}\left(G(s)\right)$$

where $G(s) = C(sI - A)^{-1} B + D$

The above fact implies that

$$\text{normal rank}\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = n + \text{normal rank}\left(G(s)\right)$$

and

$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix}$ has full column (row) rank

$\Updownarrow$

G(s) has full column (row) rank

Thus, $\forall z_o \notin \lambda(A)$,

$$\begin{bmatrix} A - z_o I & B \\ C & D \end{bmatrix} \text{ loses rank} \quad \Leftrightarrow \quad G(z_o) \text{ loses rank}$$

which indicates that the above definition of invariant zeros is the same as the definition of zeros of G(s) via SM form, when there are no common poles and zeros.

Proof of Fact:

$\forall s \notin \lambda(A), \quad (A - sI)^{-1}$ exists. Thus,

$$\begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ C(A - sI)^{-1} & I \end{bmatrix} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix}$$

$\Rightarrow$

$$\text{rank} \begin{bmatrix} A - sI & B \\ C & D \end{bmatrix} = \text{rank} \begin{bmatrix} A - sI & B \\ 0 & G(s) \end{bmatrix} = n + \text{rank}\left(G(s)\right)$$

#

The following theorem reveals the blocking properties of the invariant zeros:

Theorem 3.3.2

Let (A, B, C, D) be a realization of $G(s) \in \mathcal{R}^{p \times m}$. Then,

(a). The output due to the input $u(t) = u_o e^{\lambda t}$ and the initial state $x(0) = (\lambda I - A)^{-1} B u_o$, where u_o is an arbitrary constant vector and $\lambda \in C$ is not an eigenvalue of A, is

$$y(t) = G(\lambda)\, u_o\, e^{\lambda t}$$

(b). In particular, for full column normal rank G(s), if $\lambda = z_o$ is an invariant zero, then, with u_o and $x(0) = x_o$ chosen to be as in the above lemma,

$$y(t) = 0 \quad , \quad \forall t$$

Proof:

(a). $u(t) = u_o e^{\lambda t} \quad \Rightarrow \quad U(s) = \dfrac{1}{s-\lambda} u_o$

$$
\begin{aligned}
Y(s) &= G(s)U(s) + C(sI-A)^{-1}\, x(0) \\
&= \frac{1}{s-\lambda}\Big[C(sI-A)^{-1}B + D\Big]u_o + C(sI-A)^{-1}\,(\lambda I - A)^{-1}B\,u_o \\
&= \frac{1}{s-\lambda}\left\{\begin{aligned} &C(sI-A)^{-1}B\,u_o + D\,u_o \\ &\qquad\qquad + (s-\lambda)\,C(sI-A)^{-1}\,(\lambda I-A)^{-1}B\,u_o \end{aligned}\right\} \\
&= \frac{1}{s-\lambda}\left\{\begin{aligned} &C(sI-A)^{-1}\Big[(\lambda I-A) + (s-\lambda)I\Big](\lambda I-A)^{-1}B\,u_o \\ &\qquad\qquad + D\,u_o \end{aligned}\right\} \\
&= \frac{1}{s-\lambda}\left\{C(\lambda I-A)^{-1}B\,u_o + D\,u_o\right\} = \frac{1}{s-\lambda}G(\lambda)\,u_o
\end{aligned}
$$

$$\therefore \quad y(t) = G(\lambda)\,u_o e^{\lambda t}$$

(b). With

$$\begin{bmatrix} A - z_o I & B \\ C & D \end{bmatrix} \begin{bmatrix} x_o \\ u_o \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{cases} x_o = (z_o I - A)^{-1} B u_o \\ C x_o + D u_o = 0 \end{cases}$$

$$\Rightarrow \quad C(z_o I - A)^{-1} B u_o + D u_o = 0$$

$$\Rightarrow \quad G(z_o) u_o = 0$$

$$\therefore \quad y(t) = G(\lambda) u_o e^{\lambda t} = 0$$

#