10.1: Basic Robustness of Uncertain Systems

The previous chapter covers how to represent uncertain systems as LFTs on unknown, but structured uncertainty matrices. So the next question is:

How to analyze the robustness of these uncertain systems?
Answers are provided using the structured singular value, $\mu$. For details, refer to the paper by Packard and Doyle:


The basic idea is to reduce the test of the robustness of a MIMO feedback loop with uncertainties to a determinant condition that can be easily solved numerically via certain optimization algorithms such as the Linear Matrix Inequalities (LMI).
It is shown in the previous chapter that a MIMO feedback loop with uncertainties can be recast into the feedback loop diagram below,

where

- $M$ is the nominal closed-loop system which is stable
- $\Delta$ is a structured perturbation belong to a known structure $\Phi$ with the property that

\[ \forall \Delta \in \Phi, \ 0 \leq \tau \leq 1 \implies \tau \Delta \in \Phi \]
The closed-loop TF matrix of the above feedback loop is \((I - M\Delta)^{-1}\). So the stability problem of the above feedback loop (i.e., the robust stability of original MIMO feedback system) can be formulated as:

Given \(M(s) \in \mathcal{S}^{n \times n}\) and \(\Phi\), determine if 
\[(I - M\Delta)^{-1} \in \mathcal{S}^{n \times n}\] 
for all \(\Delta \in \Phi\).

The following theorems reduce such a question to a determinant condition. For simplicity, let us first consider the case when \(\Phi \subset \mathcal{C}^{n \times n}\). In such a case, as \(M(s) \in \mathcal{S}^{n \times n}\), then, \(\forall \Delta \in \Phi, (I - M\Delta) \in \mathcal{S}^{n \times n}\). Thus \((I - M\Delta)^{-1} \in \mathcal{S}^{n \times n}\) iff \(\det(I - M\Delta) \in \mathcal{U}_s\), a condition for which this can be checked is given by the following theorem.
Theorem 10.1

Given $\Phi$ and $M$ as above. Then
$$\det(I - M\Delta) \in \mathcal{U}_s \text{ for all } \Delta \in \Phi \text{ iff}$$

1. $\det(I - M(\infty)\Delta) \neq 0 \text{ for all } \Delta \in \Phi.$
2. $\det(I - M(j\omega)\Delta) \neq 0 \text{ for all } \omega \in \mathbb{R} \text{ and all } \Delta \in \Phi.$
Proof:

⇒

If conditions are violated, then, there exist $\Delta$ that $\det\left(I - M(s)\Delta\right) = 0$ either for $s = j\omega$ or $s = \infty$. This implies $I - M(s)\Delta$ has a pole either at $\infty$ (so $(I - M\Delta)^{-1}$ is improper) or on the imaginary axis, which contradicts with the assumption that $\det\left(I - M\Delta\right) \in U_s$ for all $\Delta \in \Phi$.

⇐

When both conditions (which is along imaginary axes) are satisfied, we need to check $\det\left(I - M(s)\Delta\right) \neq 0$ for all $\Delta \in \Phi$ everywhere in the right-half-plane as well to make sure $1/\det\left(I - M(s)\Delta\right) \in S$. For this purpose, pick any $\Delta \in \Phi$. Look at the function

$$r_\tau(s) := \det\left(I - \tau M(s)\Delta\right) \in S$$

Since $\left\{M(j\omega) : \omega \in \mathcal{R}\right\} \cup M(\infty)$ is bounded, and $M(\infty)$
exists (i.e., $M(j\omega)$ has a limit), $r_\tau(j\omega)$ is a uniformly continuous function of $\omega$ and $\tau$. Thus, noting $r_0(s) = 1$, for some $0 < \bar{\tau} < 1$,

$$|1 - r_\bar{\tau}(j\omega)| \leq 0.5$$

for all $\omega$ (including $\infty$). Hence (by Argument Principle), $r_\bar{\tau}(s)$ has no zeros in RHP as its Nyquist plot does not encircle the origin due to the above fact. Now, let $\tau$ change from $\bar{\tau}$ to 1. Noting $\tau \Delta \in \Phi$, the two conditions guarantee that there are no $\omega$ or $\tau$ that $r_\tau(j\omega)$ passes through zero. Thus, $\forall \bar{\tau} \leq \tau \leq 1$, along the Nyquist path, the encirclements of $r_\bar{\tau}(s)$ with respect to the origin remain the same as that of $r_\bar{\tau}(s)$, i.e., 0. Hence $r_1(s)$ has no zeros in RHP (or at $s=\infty$). Therefore $r_1 \in \mathcal{U}_s$ as desired.

#
The above theorem still holds if we let $\hat{\Phi}$ be dynamic, i.e.,
given $\Phi$, define

$$
\hat{\Phi} := \left\{ \hat{\Delta}(s) \in \mathcal{S}^{n \times n} : \hat{\Delta}(\infty) \in \Phi, \forall \omega, \hat{\Delta}(j\omega) \in \Phi \right\}
$$

Note that $\Phi \subset \hat{\Phi}$. Essentially the same proof gives:

**Theorem 10.2**

*Given* $\Phi$ and $M$ as above, and $\hat{\Phi}$ defined from $\Phi$.

*Then* $\det(I - M(s)\hat{\Delta}(s)) \in \mathcal{U}_s$ for all $\hat{\Delta} \in \hat{\Phi}$ *iff*

1. $\det(I - M(\infty)\Delta) \neq 0$ for all $\Delta \in \Phi$

2. $\det(I - M(j\omega)\Delta) \neq 0$ for all $\omega \in \mathbb{R}$ and all $\Delta \in \Phi$
The above theorems indicate that the robust stability boundary (i.e., the allowable degree of uncertainties for stability) can be determined by checking when the above conditions will fail, i.e., \( \det(I - M(j\omega)\Delta) \) become zero for some \( \omega \) including \( \infty \), which can be accomplished using MATLAB \( \mu \)-tools as follows.

10.2: \( \mu \) Analysis, Robust Stability

Every \( \mu \)-analysis consists of the following steps:

1. Recast the problem into the feedback loop diagram (called the *analysis diagram*) below,
where

- $M$ is a known linear system,
- $\Delta$ is a structured perturbation

2. Calculate a frequency response of $M$

3. Describe the structure of the perturbations $\Delta$.

4. Run the command $\text{mu}$ on the frequency response

$$\mu_\Delta(M) := \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0\}}$$

5. Plot the bounds obtained from the $\mu$ calculation
10.3: Analysis Diagram from System Diagram

Consider the mass/damper/spring system, $H_{mix}$, with both parametric uncertainty and unmodeled dynamics, along with feedback controller $K$, so $u = Ky$

Group $H_{mix}$ and $K$ together, namely $M := F_L(H_{mix}, K)$, to get
\[ M := F_L(H_{mix}, K) \]
10.4: Structure of Uncertainty

The structure of the perturbation matrices must be passed to the \texttt{mu} command. Three attributes about each uncertainty block must be specified:

1. The type (real parameter vs. unmodeled dynamics) of the perturbation.
2. The dimension of the perturbation.
3. The \# of independent locations that the particular uncertainty occurs (Gain/Pade/Lag example had $\delta_1$ affecting the system in 2 independent locations)

Uncertainty structure information is stored as a $n \times 2$ array (called the \textit{block structure array}) where
• $n$ is the number of different perturbation elements in the uncertainty matrix.

• The $i$-th row describes the $i$-th uncertain block, using conventions
  
  – A scalar real parameter is denoted [-1 1] (or [-1 0]).
  – A repeated (f times) real parameter is denoted [-f 0].
  – A 1x1 (i.e., scalar) unmodeled dynamics perturbation (later we refer to this as complex) is denoted [1 1].
  – A $r \times c$ (i.e., full, rectangular) unmodeled dynamics block is denoted [r c].

Ordering of the uncertainty elements is (and must be) consistent with the ordering of the input/output channels of the known systems. Use the notation $\Delta$ to represent the set of all perturbation matrices with the appropriate structure.
\[
\Delta := \begin{bmatrix}
\delta_1 & 0 & 0 & 0 \\
0 & \delta_2 & 0 & 0 \\
0 & 0 & \delta_3 & 0 \\
0 & 0 & 0 & \delta_4 \\
\end{bmatrix}
: \delta_1 \in \mathbb{R}, \delta_2 \in \mathbb{R}, \delta_3 \in \mathbb{R}, \delta_4(s)
\]

and is represented as

\[\text{>> deltaset =} \begin{bmatrix}
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
1 & 1 \\
\end{bmatrix};\]
Now that the uncertainty structure has been represented, we can compute the size of perturbations against which the system is robustly stable.

10.5: Robust Stability Tests

We need to calculate a frequency response of $M$, and then compute the structured singular value $\mu$ of $M$ with respect to the uncertainty set $\Delta$. At each frequency, the matrix $M(j\omega)$ is passed to the $\mu$ algorithm and $\mu(M(j\omega))$ is computed and then plotted. Use the notation $\mu_\Delta(M(j\omega))$, to emphasize the dependency of the function on both

- $M$, and
- on the uncertainty set $\Delta$.

Let $\beta$ denote peak (across frequency $\omega$) of $\mu_\Delta(M(j\omega))$.
\[
\max_{\omega \in \mathcal{R}} \mu_\Delta (M(j\omega)) =: \beta
\]

which is interpreted as follows:

1. For \textit{all} perturbation matrices \( \Delta \) with
   - The appropriate structure (i.e., any \( \Delta \in \Delta \)),
   - \( \max_{\omega} \bar{\sigma}[\Delta(j\omega)] < \frac{1}{\beta} \),

the perturbed system

\[ \xymatrix{ \Delta \ar[r] & M \ar[l] } \]

is stable.
2. Moreover, there is a particular perturbation matrix with
   \[ \Delta \in \Delta, \text{ and} \]
   \[ \max_{\omega} \sigma[\Delta(j\omega)] = \frac{1}{\beta}, \]
   that causes instability.

Hence, we think of

\[ \frac{1}{\max_{\omega} \mu_{\Delta}(M(j\omega))} \]

as a stability margin with respect to the structured uncertainty set \( \Delta \) affecting \( M \).
10.6: Robust Stability Test, Upper and Lower Bounds

The software does not compute $\mu$ exactly, but bounds it from above and below by several optimization steps. Let

- $\beta_u := \text{peak (across frequency) of the upper bound for $\mu$}$,
- $\beta_l := \text{peak of the lower bound for $\mu$}$.

Then

- For all perturbation matrices $\Delta \in \Delta$ satisfying

$$\max_{\omega} \bar{\sigma} [\Delta(j\omega)] < \frac{1}{\beta_u},$$

the perturbed system is stable;
• There is a particular perturbation matrix \( \Delta \in \Delta \) satisfying

\[
\max_{\omega} \bar{\sigma}[\Delta(j\omega)] = \frac{1}{\beta_l}
\]

that causes instability.

The gap between the upper and lower bounds translates into gaps between the conclusions “guaranteed robust stability” and “not robustly stable”. The destabilizing perturbation matrix (of size \(1/\beta_l\)) can be constructed from the \(\mu\) calculation using the command “\texttt{dypert}”. 
10.7: MIMO Robust Performance

Now, go beyond robustness of stability. What about robustness of performance? To answer this question, manipulate the closed-loop system into the feedback loop diagram below using LFTs,

![Feedback Loop Diagram](image)

where

- $M$ is a known linear system,
- $\Delta$ is a structured perturbation from a problem-dependent allowable uncertainty set $\Delta$, and
- $d$ and $e$ are the generalized disturbance and error that characterizes performance objective.
The performance of MIMO control systems will be characterized using $\mathcal{H}_\infty$ norms. We assume that good performance is equivalent to

$$\|T\|_\infty := \max_{\omega \in \mathbb{R}} \sigma(T(j\omega)) \leq 1$$

where $T$ is the weighted, closed-loop transfer function matrix of interest.

For robust performance of uncertain systems, $T$ is the uncertain transfer function from $d \rightarrow e$, so $T = F_U(M, \Delta)$, which a function of $\Delta$, through the elements of $M$, and the LFT formula.

**Question:**

How big can the transfer function $F_U(M, \Delta)$ get as $\Delta$ takes on its allowed values?
Precisely: Given $M$ and $\Delta$, the LFT

is said to achieve **Robust Performance** if for all perturbations of

- $\Delta \in \Delta$, and
- $\max_{\omega} \bar{\sigma}(\Delta(j\omega)) < 1$,

the LFT is stable, and has $\left\| F_U(M, \Delta) \right\|_\infty \leq 1$. 
10.8: Robust Performance Test using $\mu$

How can $\mu$ be used to assess robust performance?

Main idea—relate the size of a transfer function to a robust stability test.

Suppose that $T$ is a given, stable system, with input dimension $n_d$ and output dimension $n_e$. By the Nyquist and small-gain theorem, we know that $\|T\|_\infty \leq 1$ iff the feedback loop shown below is stable for every stable $\Delta_F(s)$ (of dimension $n_d \times n_e$) satisfying $\|\Delta_F\|_\infty < 1$.

Stable for all $\|\Delta_F\|_\infty < 1$
Hence, a transfer function $T$ is small, i.e., $\|T\|_{\infty} \leq 1$, if and only if $T$ can tolerate all possible stable feedback perturbations without leading to instability:

The size of a transfer function can be determined using a robust stability test!

This ultimately allows us to pose the robust performance question as a robust stability question as follows. For robust performance problems, transfer function in question is an LFT,

$$T = F_U(M, \Delta)$$

\[
\begin{array}{c}
\Delta \\
M \\
e \\
d
\end{array}
\]
Following the argument presented, it follows that

$$\left\| F_U (M, \Delta) \right\|_{\infty} \leq 1$$

for all perturbations $\Delta \in \Delta$ satisfying $\max_{\omega} \bar{\sigma}[\Delta(j\omega)] < 1$

iff the LFT below is stable for all $\Delta \in \Delta$ and all stable $\Delta_F(s)$ satisfying

$$\max_{\omega} \bar{\sigma}[\Delta(j\omega)] < 1 \quad \text{and} \quad \max_{\omega} \bar{\sigma}[\Delta_F(j\omega)] < 1$$
But this is exactly a Robust Stability problem for $M$, subjected to perturbation matrices of the form

$$\Delta_P = \begin{bmatrix} \Delta & 0 \\ 0 & \Delta_F \end{bmatrix}$$

where $\Delta_F$ is a full unmodeled dynamics block. Hence, we use robust stability techniques – on a larger problem, computing $\mu_{\Delta_P}(M(j\omega))$ to determine bounds on robust performance for our original problem!

IMPORTANT

Robust Performance is characterized by introducing a fictitious uncertainty block across the disturbance/error channels and carrying out a Robust Stability Analysis.
In summary, each robust performance $\mu$-analysis consists of the following steps:

1. Recast the problem into the feedback loop below

```
          \[\Delta\]

  \[M\]  \[d\]  \[e\]
```

where $M$ is a known linear system, and $\Delta \in \Delta$ is a structure perturbation, and $d$ and $e$ are the generalized disturbance and error that characterize the performance objective.

2. Calculate a frequency response of $M$

3. Describe the structure of perturbation set $\Delta$. 
4. Use the dimensions of the disturbance/error channels to define a fictitious uncertainty block $\Delta_F$, and augment this with the actual uncertainty structure of $\Delta$ giving an extended uncertainty set $\Delta_p$, where $\Delta_F$ is a full, unmodeled dynamics block of dimension $n_d \times n_e$.

\[
\Delta_p := \begin{bmatrix}
\Delta & 0 \\
0 & \Delta_F
\end{bmatrix}
\]
5. Compute $\mu_{\Delta_p}(M(j\omega))$ on the frequency response of $M$, using the augmented uncertainty set $\Delta_p$.

6. Plot the bounds obtained from the $\mu$ calculation. Let $\beta$ denote the peak of the $\mu$-plot

$$\max_{\omega \in \mathbb{R}} \mu_{\Delta_p}(M(j\omega)) =: \beta$$

Then,

a) For all perturbation matrices satisfying $\Delta \in \Delta$ and

$$\max_{\omega} \bar{\sigma}[\Delta(j\omega)] < \frac{1}{\beta},$$

the perturbed system is stable and

$$\|F_U(M, \Delta)\|_{\infty} \leq \beta$$
b) Moreover, there is a particular perturbation satisfying $\Delta \in \Delta$ and

$$\max_{\omega} \sigma[\Delta(j\omega)] = \frac{1}{\beta},$$

that causes either $\left\| F_U(M, \Delta) \right\|_\infty = \beta$, or instability.

Recall, exact computation of $\mu$ is not possible, so what implications are true using the bounds.

10.9: Robust Performance Tests, Bound

Let

- $\beta_u := \text{peak (across frequency) of the upper bound for } \mu_{\Delta_p}(M(j\omega))$
- $\beta_l := \text{peak of the lower bound for } \mu_{\Delta_p}(M(j\omega))$
Then

a) For all perturbation matrices satisfying $\Delta \in \Delta$ and

$$\max_{\omega} \sigma[\Delta(j\omega)] < \frac{1}{\beta_u},$$

the perturbed system is stable and

$$\|F_U(M, \Delta)\|_{\infty} \leq \beta_u$$

b) Moreover, there is a particular perturbation satisfying $\Delta \in \Delta$ and

$$\max_{\omega} \sigma[\Delta(j\omega)] = \frac{1}{\beta_l},$$

that causes either $\|F_U(M, \Delta)\|_{\infty} \geq \beta_l$, or instability.

Hence the gap between the upper and lower bounds leads to gaps in the inability to precisely determine the robust performance.
10.10: MATLAB Mu-Tools

\[
\text{bounds} = \text{mussv}(M, \text{BlockStructure})
\]

calculates upper and lower bounds on the structured singular value, or \( \mu \), for a given block structure. \( M \) is a double array, an frd model, or a state-space (ss) model.

\[
[\text{stabmarg}, \text{wcu}] = \text{robstab}(\text{usys})
\]
\[
[\text{stabmarg}, \text{wcu}, \text{info}] = \text{robstab}(\text{usys})
\]

calculates the robust stability margin for an uncertain system. This stability margin is relative to the uncertainty level specified in \( \text{usys} \). A robust stability margin greater than 1 means that the system is stable for all values of its modeled uncertainty. A robust stability margin less
than 1 means that the system becomes unstable for some values of the uncertain elements within their specified ranges. For example, a margin of 0.5 implies the following:

- **usys** remains stable as long as the uncertain element values stay within 0.5 normalized units of their nominal values.

- There is a destabilizing perturbation of size 0.5 normalized units.

The structure **stabmarg** contains upper and lower bounds on the actual stability margin and the critical frequency at which the stability margin is smallest. The structure **wcu** contains the destabilizing values of the uncertain elements.
[perfmarg, wcu] = \texttt{robgain}(\texttt{usys}, \texttt{gamma})
calculates the robust performance margin for an uncertain system and the performance level gamma. The performance of \texttt{usys} is measured by its peak gain or peak singular value (see Robustness and Worst-Case Analysis). The performance margin is relative to the uncertainty level specified in \texttt{usys}. A margin greater than 1 means that the gain of \texttt{usys} remains below gamma for all values of the uncertainty modeled in \texttt{usys}. A margin less than 1 means that at some frequency, the gain of \texttt{usys} exceeds gamma for some values of the uncertain elements within their specified ranges. For example, a margin of 0.5 implies the following:
• The gain of \texttt{usys} remains below gamma as long as the uncertain element values stay within 0.5 normalized units of their nominal values.

• There is a perturbation of size 0.5 normalized units that drives the peak gain to the level gamma.

The structure \texttt{perfmarg} contains upper and lower bounds on the actual performance margin and the critical frequency at which the margin upper bound is smallest. The structure \texttt{wcu} contains the uncertain-element values that drive the peak gain to the level gamma.
\[ [K, CL_{\text{perf}}] = \text{musyn}(P, \text{nmeas}, \text{ncont}) \]

designs a robust controller for an uncertain plant using D-K iteration, which combines $H_\infty$ synthesis (K step) with $\mu$ analysis (D step) to optimize closed-loop robust performance.

You can use \texttt{musyn} to:

- Synthesize "black box" unstructured robust controllers.
- Robustly tune a fixed-order or fixed-structure controller made up of tunable components such as PID controllers, state-space models, and static gains.

For additional information about performing $\mu$ synthesis and interpreting results, see Robust Controller Design Using Mu Synthesis.