Analysis of Discrete-Time Systems

- **Stability**
  - Lyapunov Stability
  - Bounded-Input-Bounded-Output (BIBO) Stability
  - Five methods to determine the stability of discrete LTI system (other than eigenvalues of state matrix):
    - Lyapunov's second method.
    - Bilinear transformation and RH criterion.
    - Jury's stability criterion in z domain.
  - Root Locus
  - Nyquist criterion

- **Sensitivity and Robustness**

- **Controllability and Observability**
Stability

- For nonlinear dynamic systems, stability is a **local characteristics** associated with an **equilibrium point (state)**
  - The system may have multiple isolated equilibrium points or no equilibrium point at all; some of the equilibrium points might be stable -- solutions starting nearby of the equilibrium point converge to the equilibrium point -- while others unstable (i.e., solutions starting nearby of the equilibrium point diverge).

- For linear systems, stability is unique for the system
  - The system has either only one equilibrium point at zero or infinite non-isolated multiple equilibrium points. In either case, the solutions for all initial conditions either converge to zero (stable), or diverge (unstable), or neither converge or diverge (marginally stable).
Equilibrium State (Point)

Definition: A constant state vector $x_E \in \mathbb{R}^n$ is said to be an equilibrium state (point) of the system

Continuous-Time
$\dot{x}(t) = F \cdot x(t)$

Discrete-Time
$x(k + 1) = A \cdot x(k)$

If it satisfies the following condition:

$F \cdot x_E = 0$

$A \cdot x_E = x_E$

Observation:
- Apparently, the origin, i.e., $x = 0$, is an equilibrium state for both the continuous-time and the discrete-time system.
- $x = 0$ is the only equilibrium state if the system matrix $F$ (or $A$) does not have eigenvalues 0 (or 1 for DT system).
- The eigenvectors corresponding to the 0 (or 1 for DT system) eigenvalues are the non-zero equilibrium states of the system.
Equilibrium States (Points)

- **Example**: Find the equilibrium states of the system

\[ x(k+1) = A \cdot x(k), \quad \text{where} \quad A = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} \]

To find the equilibrium states, we can use the definition

\[ x_E = A \cdot x_E \quad \Rightarrow \quad (1 \cdot I - A) \cdot x_E = 0 \quad \Rightarrow \quad \begin{bmatrix} 1/2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \Rightarrow x_{1E} = 2 \cdot x_{2E} \]

*Note that the above equation implies that the non-zero equilibrium states are in the eigenvector direction \([2 \quad 1]\), that is corresponding to the eigenvalue 1.*

all the states (points) that satisfy the relationship \( x_1 = 2x_2 \) are equilibrium states for this system.
### State Trajectory

- If the initial condition of the system is on the equilibrium states, the system states will remain on the equilibrium states for all time.

- If the initial states are away from the equilibrium states, the states will evolve along state trajectories either moving away from the equilibrium (unstable), or stays nearby all the time (if neither diverge nor converge, then, stable in the sense of Lyapunov, and if eventually converge to the equilibrium, then asymptotically stable in the sense of Lyapunov).
**State Trajectory**

- **Example:** Find the state trajectory of the system

\[
x(k + 1) = A \cdot x(k) = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \quad \Rightarrow \quad x(k) = A^k \cdot x(0)
\]

\[
\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = \begin{bmatrix} (1/2)^k \cdot x_1(0) + \sum_{j=0}^{k-1} (1/2)^j \cdot x_2(0) \\ x_2(0) \end{bmatrix} \quad \xrightarrow{k \to \infty} \quad \begin{bmatrix} 2x_2(0) \\ x_2(0) \end{bmatrix}
\]

*Phase Plane Plot*

- State trajectory
- Equilibrium State

\[x_1 = 2x_2\]
Phase Plane Plot

% Define Continuous-time system
F=[0 1; -1 -1]; G=[0;1]; C=eye(2); D=[0;0];
Gp=ss(F,G,C,D);
y=initial(Gp,[0;1],0:0.01:10);
xc_1=y(:,1); xc_2=y(:,2);

Gd1=c2d(Gp,0.1,'zoh'); % 10 Hz sample
Gd2=c2d(Gp,1,'zoh'); % 1 Hz Sample

yd1=initial(Gd1,[0;1],0:0.1:10);
xd1_1=yd1(:,1); xd1_2=yd1(:,2);

yd2=initial(Gd2,[0;1],0:1:10);
xd2_1=yd2(:,1); xd2_2=yd2(:,2);

plot(xc_1,xc_2,'b-',xd1_1,xd1_2,'go', xd2_1,xd2_2,'rd');
Stability of Equilibrium Points
(in the sense of Lyapunov)

Definition: An equilibrium point of a system is said to be stable
(in the sense of Lyapunov, isL), if for every $\varepsilon > 0$, there exists a

Continuous-Time

$\delta(\varepsilon, t_0) > 0$

such that for every initial condition $x(t_0)$ that satisfies

$$\|x(t_0) - x_E\| < \delta(\varepsilon, t_0)$$

then

$$\|x(t) - x_E\| < \varepsilon, \quad \forall \ t > t_0$$

Discrete-Time

$\delta(\varepsilon, k_0) > 0$

For LTI systems, all poles have negative real parts (stable LTI system) or simple poles on imaginary axis only (marginally stable LTI system).

For LTI systems, all poles inside unit circle (stable LTI DT system) or simple poles on unit circle only (marginally stable LTI DT system).
Observations:

- Stability is a local concept associated with the particular equilibrium point in consideration.
- If $\delta$ in the definition does not depend on the initial time, then the equilibrium point is said to be uniformly stable.
Asymptotic Stability of Equilibrium Points
(in the sense of Lyapunov)

Definition: An equilibrium state $x_E$ of a system is said to be asymptotically stable (A.S.) if

1. It is stable (in the sense of Lyapunov).
2. Every $x(t_0)$ or $x(k_0)$ such that

   \[
   \|x(t_0) - x_E\| < \delta(\epsilon, t_0)
   \]

   Continuous-Time

   \[
   \|x(k_0) - x_E\| < \delta(\epsilon, k_0)
   \]

   Discrete-Time

will lead to

\[
\lim_{t \to \infty} x(t) \to x_E
\]

\[
\lim_{k \to \infty} x(k) \to x_E
\]

For LTI systems, all poles in LHP (stable LTI system)

For LTI DT systems, all poles inside unit circle (stable LTI DT system)
Asymptotic and Exponential Stability

- The largest region (neighborhood) that the resulting solutions converge to an asymptotic stable equilibrium point is called the domain of attraction of the equilibrium point.

**Definition**: If \( x_E \) is asymptotically stable for any \( \delta > 0 \), then \( x_E \) is said to be globally asymptotically stable (A.S. in the large).

**Definition**: \( x_E \) is exponentially stable, if (1) It is stable, and (2) There exists a \( \delta > 0 \), such that, for every

**Continuous-Time**
\[
\|x(t_0) - x_E\| < \delta
\]

there exists a constant \( 0 < M < \infty \), and a constant \( \alpha \), such that

\[
\|x(t) - x_E\| < M \cdot e^{-\alpha (t-t_0)} \|x(t_0) - x_E\|
\]

**Discrete-Time**
\[
\|x(k_0) - x_E\| < \delta
\]

\[
\|x(k) - x_E\| < M \cdot \alpha^{(k-k_0)} \|x(k_0) - x_E\|
\]
Stability

Observations:

- For nonlinear systems, stability is a property associated with a specific equilibrium state; a nonlinear system could have both stable and unstable equilibrium points. However, for linear systems, stability is unique to the system, as all the equilibrium points (if more than one) have the same stability properties.

- For LTI systems,
  
  Stability (isL) is equivalent to (⇔) uniform stability
  Asymptotic stability is equivalent to (⇔) exponential stability

- Stability is a system property that has nothing to do with the input of the system. Hence stability definitions are for free responses of a system.
Input-Output Stability of a System

- **Bounded-Input Bounded-Output (BIBO) Stability**
  A system is said to be BIBO stable if all bounded inputs result in bounded outputs for all initial conditions.

- An LTI system can be stable in the sense of Lyapunov but not BIBO stable. However, asymptotic stability of an LTI system does imply the BIBO stability of the system.
Stability is L and BIBO Stability

Example: (Stable but not BIBO stable)

\[
x(k+1) = \begin{bmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{bmatrix} \cdot x(k) + \begin{bmatrix} 1 - \cos(\omega T) \\ \sin(\omega T) \end{bmatrix} \cdot u(k)
\]

\[
y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot x(k)
\]

\[
\det \begin{bmatrix} \lambda - \cos(\omega T) & -\sin(\omega T) \\ \sin(\omega T) & \lambda - \cos(\omega T) \end{bmatrix} = 0 \quad \Rightarrow \quad \lambda = \cos(\omega T) \pm j \sin(\omega T) = e^{\pm j\omega T}
\]

Stable but not asymptotically stable!
Lyapunov Stability Theorems

**Definition: Positive Definite Function (PDF)**
A scalar function $V(x)$ is said to be *positive definite* in a region $V$ that includes the origin (equilibrium state), if

(1) $V(x) > 0$, for all $x \neq 0$
(2) $V(x) = 0$, when $x = 0$

**Definition: Positive Semi-Definite Function (PSDF)**
A scalar function $V(x)$ is said to be *positive semi-definite*, if $V(x) = 0$ at the origin (equilibrium state) and, at all other points, $V(x) \geq 0$

**Definition: Negative Definite Function (NDF)**
A scalar function $V(x)$ is said to be *negative definite*, if $-V(x)$ is PDF.
Lyapunov Stability Theorems

Definition: Lyapunov Function
A continuously differentiable function $V(x)$ is called a Lyapunov Function for the following autonomous (or time-invariant) system

Continuous-Time
\[ \dot{x}(t) = f(x(t)), \quad f(0) = 0 \]

if

1. $V(0) = 0$
2. $V(x)$ is PDF
3. $\dot{V}(x) = \frac{\partial}{\partial x} V(x) \cdot \dot{x}$

\[ = \frac{\partial}{\partial x} V(x) \cdot f(x) \]

is NDF

Discrete-Time
\[ x(k+1) = f(x(k)), \quad f(0) = 0 \]

or

\[ \Delta V(x) = V(x(k+1)) - V(x(k)) \]

or

\[ = V(f(x(k))) - V(x(k)) \]

\[ = V(f(x)) - V(x) \]
Lyapunov Stability Theorems

- Interpretation of the Lyapunov Function

\[ V(x) = C_3 \]
\[ V(x) = C_2 \]
\[ V(x) = C_1 \]

\[ C_3 > C_2 > C_1 \]

- Use Lyapunov function for controller synthesis
  Design a control law (algorithm) to make the generalized energy function, \( V(x) \), into a Lyapunov function.
Lyapunov Stability Theorems

- Lyapunov Stability Theorem
  Given a system described by

  **Continuous-Time**
  \[ \dot{x}(t) = f(x(t)), \quad f(0) = 0 \]

  **Discrete-Time**
  \[ x(k+1) = f(x(k)), \quad f(0) = 0 \]

  If there exists a continuous differentiable PDF \( V(x) \) such that

  \[
  \dot{V}(x) = \frac{\partial}{\partial x} V(x) \cdot \dot{x}
  \]

  or

  \[
  \Delta V(x) = V(x(k+1)) - V(x(k))
  \]

  \[
  \Delta V(x) = V(f(x(k))) - V(x(k))
  \]

  or

  \[
  = V(f(x)) - V(x)
  \]

  (1) If \( W(x) \) is NSDF, then, the equilibrium state is stable isL

  (1) If \( W(x) \) is NDF, then, the equilibrium state is A.S. In addition, if \( ||x|| \to \infty \) then \( V(x) \to \infty \), the equilibrium state at the origin is globally A.S.
Lyapunov Stability

- **Observations:**

- The Lyapunov method only provides a sufficient condition for stability. If we cannot find a scalar PD function \( V(x) \) of all the states that satisfies the conditions of the Theorem, it does not mean that the equilibrium is not stable.

- On the other hand, this is also very powerful. As long as one can find a Lyapunov function \( V(x) \), then, stability of the corresponding equilibrium state is proved.

- The approach can also be used for synthesizing controllers – find controls that make a PD function satisfying corresponding conditions of Lyapunov theorems to guarantee stability of the CL equilibrium, a very widely used method for nonlinear controls.
Lyapunov Stability for LTI Systems

The following statements are equivalent

**Continuous-Time Systems**

- The equilibrium state $0$ of the $n$th order system $\dot{x}(t) = A \cdot x(t)$ is *Globally Asymptotically Stable*.

- All eigenvalues of $A$ have negative real parts, i.e., $\text{Re}[\lambda(A)] < 0$.

- For any positive definite (PD) symmetric matrix $Q$, there exists a unique PD symmetric matrix $P$ which is the solution to the following *Lyapunov Equation*:

\[
PA + A^T P = -Q
\]

where $P$ is PD. $\Rightarrow V(x) = x^T P x$ is a PD function of $x$. &

\[
\dot{V} = x^T P \dot{x} + x^T P x = x^T A^T P x + x^T P A x = x^T (A^T P + PA) x = -x^T Q x
\]

is a PD function of $x$, since $Q > 0$.

**Discrete-Time Systems**

- The equilibrium state $0$ of the $n$th order system $x(k + 1) = A \cdot x(k)$ is *Globally Asymptotically Stable*.

- All eigenvalues of $A$ have magnitude less than 1, i.e., $\|\lambda(A)\| < 1$.

- For any positive definite (PD) symmetric matrix $Q$, there exists a unique PD symmetric matrix $P$ which is the solution to the following *Discrete-Time Lyapunov Equation*:

\[
A^T PA - P = -Q
\]
Lyapunov Stability for LTI Systems

*Observations:*

- It is convenient to choose $Q = I$ to solve the Lyapunov equation for $P$. Check the positive definiteness of $P$ to judge the system stability.

- If $P$ is positive definite, then the system is globally asymptotically stable, otherwise, the system is unstable.

- The MATLAB function `lyap()` and `dlyap()` can be used to solve the continuous-time and discrete-time Lyapunov equations.
Positive Definite Matrix

Definition: Positive Definite Matrix (PDM)
A matrix $P$ is said to be positive definite if $x^T P x > 0$ for all $x \neq 0$.

- Necessary and sufficient (if and only if, iff) conditions for a real symmetric PD matrix $P$:
  - All the eigenvalue of $P$ are positive, i.e., $\lambda(A) > 0$
  - All the principal minors (determinant of the upper $k \times k$ matrix, $k = 1, \ldots, n$) of $P$ are positive.
  - There exists a nonsingular matrix $W$ such that $P = W^T W$
Lyapunov Stability

**Example:** Check the stability of the following system using Lyapunov’s second method

\[
x(k + 1) = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} \cdot x(k)
\]

The discrete-time Lyapunov equation is

\[
A^T PA - P = -Q
\]

By letting \( Q \) equals to the identity matrix, we have

\[
\begin{bmatrix} 0 & -0.5 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}
\]

Use \texttt{dlyap()}, or solve the 3 equations directly, we have

\[
P = \begin{bmatrix} 11/5 & 8/5 \\ 8/5 & 24/5 \end{bmatrix}
\]

\[
\det(p_{11}) = \frac{11}{5} > 0,
\]

\[
\det \begin{bmatrix} 11/5 & 8/5 \\ 8/5 & 24/5 \end{bmatrix} = 8 > 0
\]

\[
\Rightarrow \text{P is positive definite, the system is A.S.}
\]
Stability Analysis Methods for Discrete-Time LTI system

- Lyapunov's second method (done!)
- Bilinear transformation to continuous-time and use Routh-Hurwitz criterion
- Jury's stability criterion in the z domain
- Nyquist criterion
The bilinear transformation maps the inside of the unit circle to (and from) the open left half plane.

The characteristic equation of the discrete-time system in $z$ has all its roots inside the unit circle if and only if all the roots of the "transformed" characteristic equation in $s$ are in the left-half-plane (LHP) of the $s$-plane, which can be checked via Routh test.

\[ z = \frac{1+s}{1-s} \]

\[ s = \frac{z-1}{z+1} \]
If all the elements in the first column have the same sign, the roots are all stable (left half of s-plane). # of sign changes indicates # of unstable roots.
Bilinear Transform

**Example**: A discrete-time system has the characteristic equation

\[ z^3 + 0.5z^2 + 0.4z + k = 0 \]

Find the range of the parameter \( k \) for stability.

Apply bilinear transformation:

\[
\frac{1 + s}{1 - s} = z
\]

\[
(1 + s)^3 + 0.5(1 + s)^2(1 - s) + 0.4(1 + s)(1 - s)^2 + k(1 - s)^3 = 0
\]

\[
\Rightarrow (0.9 - k)s^3 + (2.1 + 3k)s^2 + (3.1 - 3k)s + (1.9 + k) = 0
\]

| \( s^3 \) | \( 0.9 - k \) | \( 3.1 - 3k \) | \( \Rightarrow k < 0.9 \) |
| \( s^2 \) | \( 2.1 + 3k \) | \( 1.9 + k \) | \( \Rightarrow k > -0.7 \) |
| \( s \) | \( 3.1 - 3k - \frac{(0.9 - k)(1.9 + k)}{2.1 + 3k} \) | \( 0 \) | \( \Rightarrow -0.564 < k < 1.064 \) |
| \( 1 \) | \( 1.9 + k \) | | \( \Rightarrow k > -1.9 \) |

\[
\Rightarrow -0.564 < k < 0.9
\]
Jury’s Stability Criterion

- The discrete-time version of the Routh-Hurwitz criterion
- It is useful for determining if the following equation has all of its zeros (roots) inside the unit circle:

\[ A(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \]

**Jury Stability Table:**

<table>
<thead>
<tr>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( \cdots )</th>
<th>( a_{n-1} )</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-)</td>
<td>( a_n )</td>
<td>( a_{n-1} )</td>
<td>( \cdots )</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>( \times \alpha_n = \frac{a_n}{a_0} )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( a_0 - \frac{a_n^2}{a_0} )</th>
<th>( a_1 - \frac{a_n a_{n-1}}{a_0} )</th>
<th>( \cdots )</th>
<th>( a_{n-1} - \frac{a_n a_1}{a_0} )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-)</td>
<td>( a_{n-1} - \frac{a_n a_1}{a_0} )</td>
<td>( a_{n-2} - \frac{a_n a_2}{a_0} )</td>
<td>( \cdots )</td>
<td>( a_0 - \frac{a_n^2}{a_0} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \frac{(a_0^2 - a_n^2)^2}{a_0(a_0^2 - a_n^2)} - (a_0 a_{n-1} - a_n a_1)^2 )</th>
<th>( \cdots )</th>
<th>( 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vdots )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Jury’s Stability Criterion

- The first and second rows are the coefficients of the characteristic equation in forward and reverse order, respectively.

- The third row is obtained to eliminate the last element in rows 1 and 2. This process was repeated until the last row consists of only one element.

- If $a_0 > 0$, then the characteristic equation has all roots inside the unit circle if and only if all the highlighted terms (leading terms of the odd rows) are positive.

- If none of the highlighted terms is zero, then the number of negative terms is equal to the number of roots outside the unit circle.
Jury’s Stability Criterion

**Example**: Use the Jury’s stability criterion to check the stability of the following characteristic equation:

\[ z^2 + 0.7z + 0.1 = 0 \]

Construct Jury’s stability table:

\[
\begin{array}{ccc}
1 & 0.7 & 0.1 \\
- & 0.1 & 0.7 & 1 & \times 0.1 \\
\hline
0.99 & 0.63 & 0 \\
- & 0.63 & 0.99 & \times 0.63/0.99 \\
\hline
0.5891
\end{array}
\]

\[ \Rightarrow \text{ The characteristic equation is stable (Hurwitz) } \]
**Root Locus Method**

### Continuous-time

![Continuous-time diagram](image)

Characteristic equation:

\[ d(s) + K \cdot n(s) = 0 \quad \Rightarrow \quad 1 + K \frac{n(s)}{d(s)} = 0 \]

### Discrete-time

![Discrete-time diagram](image)

Characteristic equation:

\[ d(z) + K \cdot n(z) = 0 \quad \Rightarrow \quad 1 + K \frac{n(z)}{d(z)} = 0 \]
Root Locus Method

- The root locus method for CT systems can be used for DT systems.
- The only change is that the stability boundary is changed from the imaginary axis to the unit circle.
- The rules for drawing the root locus are all the same.

**Example**: loop transfer function

\[ L(z) = \frac{Kz}{(z-1)(z-0.5)} \]

For \( K = 3 \):

\[ 1 + L(z) = \frac{1}{(2-z)(2-0.5)} = 0 \]

Stable when \( 0 < K < 3 \)

Can be verified using Jury’s test!
### Stability of \( \frac{Kz}{(z - 1)(z - 0.5)} \)

Characteristic equation: \( z^2 + (K - 1.5)z + 0.5 = 0 \)

\[
\begin{array}{ccc}
1 & K - 1.5 & 0.5 \\

0.5 & K - 1.5 & 1 \\
0.75 & 0.5(K - 1.5) & 0 \\
0.5(K - 1.5) & 0.75 & \times 0.5(K - 1.5)/0.75
\end{array}
\]

\[
0.75 - \frac{1}{3}(K - 1.5)^2
\]

System is on the margin of becoming unstable when

\[
0.75 - \frac{1}{3}(K - 1.5)^2 = 0 \quad \Rightarrow \quad \text{Stable when } 0 < K < 3
\]
Nyquist Stability Criterion

Nyquist Criterion for Continuous-Time LTI Systems

Principle of Argument

Let \( G(s) \) be a rational function of \( s \), with finite numbers of singularities (poles). If a contour \( C \) on the \( s \)-plane encircles \( Z \) zeros and \( P \) poles of \( G(s) \) in the clockwise (CW) (counterclockwise, CCW) direction and does not pass through any poles or zeros of \( G(s) \), then the corresponding contour \( C_L \) on the \( G(s) \) plane encircles the origin of the \( G(s) \) plane \( N = Z - P \) times in the CW (CCW) direction. The contour \( C_L \) is the plot of the values of \( G(s) \) evaluated along the contour \( C \) in the complex plane.
Nyquist Stability Criterion

- **Principle of Argument (Cauchy’s Theorem)**
  Let $L(z)$ be a rational function of $z$, with finite numbers of singularities (poles). If a contour $\mathcal{C}$ on the $z$-plane encircles $Z$ zeros and $P$ poles of $L(z)$ in the clockwise (CW) (counterclockwise, CCW) direction and does not pass through any poles or zeros of $L(z)$, then the corresponding contour $\mathcal{C}_L$ on the $L(z)$ plane encircles the origin of the $L(z)$ plane $N = Z - P$ times in the CW (CCW) direction. The contour $\mathcal{C}_L$ is the plot of the values of $L(z)$ evaluated along the contour $\mathcal{C}$ in the complex plane.
Nyquist Stability Criterion

Example: (Principle of Argument)

\[ L(z) = \frac{z}{z + 2} \]

\[ z = -1 + \alpha j, \quad \alpha \in (-1, 1) \]

\[ L(z) = \frac{-1 + \alpha j}{-1 + \alpha j + 2} = \frac{-1 + \alpha j}{1 + \alpha^2} = \frac{-1 - \alpha j}{1 + \alpha^2} = \frac{-1 - \alpha j}{1 + \alpha^2} + \frac{2\alpha}{1 + \alpha^2} \]

Contour \( \mathcal{C} \)

\( Z = 1 \) and \( P = 0 \)

\[ \Rightarrow N = Z - P = 1 \]
Nyquist Stability Criterion

Consider a unity feedback system

\[ Y(z) \]  
\[ \text{Plant} \ G(z) \]  
\[ U(z) \]  
\[ \text{Controller} \ C(z) \]  
\[ E(z) \]  
\[ R(z) \]

Closed-Loop Characteristic Equation (CLCE)

\[ 1 + G(z)C(z) = 1 + L(z) = 0 \]

\[ 1 + \frac{N_L(s)}{D_L(s)} = 0 \]

\[ \frac{N_L(s)}{D_L(s)} = L(z) = G(z)C(z) \]

is the open-loop transfer function or some time referred to as the loop transfer function (LTF)

- Zeros (roots) of \( 1 + L(z) \) \( \Leftrightarrow \) Closed-loop poles
- Poles of \( 1 + L(z) \) \( \Leftrightarrow \) Open-loop poles

Apply the POA to \( 1 + L(z) \):

Choose the Nyquist contour \( C \) to encircle all the unstable region on the \( z \) plane, then the corresponding contour on the \( 1+L(z) \) plane should encircle the origin of the \( 1+L(z) \) plane \( N = Z – P \) times.
Discrete-Time Nyquist Stability

\[ N = Z - P \]

\( N \) = Number of times \( 1 + L(z) \) encircles the origin

\( = \) Number of times \( L(z) \) encircles the \((-1, 0j)\) point

\( Z \) = Number of zeros of \( 1 + L(z) \) outside the unit disc on the \( z \) plane

\( = \) Number of unstable closed-loop poles of the unity FB system

\( P \) = Number of poles of \( 1 + L(z) \) outside the unit disc on the \( z \) plane

\( = \) Number of poles of \( L(z) \) outside the unit disc on the \( z \) plane

\( = \) Number of unstable open-loop poles of the unity FB system
Discrete-Time Nyquist Stability

Observations:

- For a unity feedback system, the closed-loop system is stable if the Nyquist plot of the LTF $L(z)$ evaluated along the contour $\mathcal{C}$ encircles the $-1$ point as many times in the CW direction as the number of unstable open-loop poles of $L(z)$

- If the open-loop system is stable, i.e., $P = 0$, then the stability of the closed-loop system is ensured if the Nyquist plot of $L(z)$ does not encircle the $-1$ point

- If $L(z) \to 0$ as $z \to \infty$, i.e., $L(z)$ is strictly proper, then parallel segments IV & VI do not influence the stability test and it is sufficient to plot the Nyquist plot of $L(z)$ evaluated on the unit circle (segments I & II) and the small semicircle at $z = 1$ (segments III & VII)
Discrete-Time Nyquist Stability

**Example:** Use the Nyquist stability criterion to find the closed-loop stability limit for the parameter $K$ for the following loop transfer function

$$L(z) = \frac{0.25K}{(z - 1)(z - 0.5)} = K \cdot \frac{0.25}{L_1(z)}$$

$$L_1(z) = \frac{0.25}{(Re^{j\omega} - 1)(Re^{j\omega} - 0.5)}$$

$\Rightarrow$ Closed-loop system will be stable if $K < 2$
Some more observations:

- The more important part of the Nyquist plot is the portion that corresponds to segments I & II of the contour $C$, which corresponds to plotting $L(e^{j\omega T})$, $\omega T \in [0, \pi]$. This is essentially the frequency response of the loop transfer function. From this point-of-view, the Bode plot and the Nyquist plot conveys the same information about the system.
Relative Stability

- **Gain Margin (GM)**
  If $\omega_P$ is the smallest phase crossover frequency such that
  \[
  \arg \left( L(e^{j\omega_P}) \right) = \angle L(e^{j\omega_P}) = -\pi \text{ or } -180^\circ
  \]
  The *gain margin* is defined to be the inverse of the LTF gain at the phase crossover frequency
  \[
  GM = \frac{1}{|L(e^{j\omega_P})|}
  \]

- **Phase Margin (PM)**
  If $\omega_G$ is the smallest gain crossover frequency such that
  \[
  \left| L(e^{j\omega_G}) \right| = 1
  \]
  The *phase margin* is defined to be
  \[
  PM = \pi + \arg \left( L(e^{j\omega_G}) \right) \text{ or } 180^\circ + \angle L(e^{j\omega_G})
  \]
Gain Margin and Phase Margin

Relative Stability

Example:

\[ G_p(s) = \frac{1}{s^2 + 1.3s + 1} \]

\[ p = -0.65 \pm 0.76i \]

PM: 134° at 0.56 rad/s

ZOH equivalent plant (T=0.4):

\[ P(z) = \frac{0.0669z + 0.05622}{z^2 - 1.471z + 0.5945} \]

\[ z = -0.84 \quad p = 0.74 \pm 0.23i \]

GM: 17.2 dB at 2.6 rad/s

PM: 129° at 0.55 rad/s

Nyquist Diagram

Bode Diagram
The sensitivity function is also the closed-loop transfer function between the reference input \( r \) and the error \( e \) as well as the negative of transfer function between the measurement noise \( n \) and the error \( e \).
Nominal Performance

Since
\[ E(z) = G_{ER}(z) \cdot R(z) = \frac{1}{1 + L(z)} \cdot R(z) = S(z) \cdot R(z) \]

\[ E(e^{j\omega T}) = G_{ER}(e^{j\omega T}) \cdot R(e^{j\omega T}) \]

\[ \Rightarrow \quad \|E(e^{j\omega T})\| \leq \|G_{ER}(e^{j\omega T})\| \cdot \|R(e^{j\omega T})\| \]

\[ \Rightarrow \quad \|E(e^{j\omega T})\| / \|R(e^{j\omega T})\| \leq \|S(e^{j\omega T})\| \]

Let the desired performance specification be
\[ \|E(e^{j\omega T})\| \leq \frac{1}{\|W_1(\omega)\|} \cdot \|R(e^{j\omega T})\| \]

\[ \Rightarrow \quad \|W_1(\omega)S(e^{j\omega T})\| < 1 \]
How much uncertainty (differences) between the plant model and the actual plant can a controller designed based on the plant model handle before the actual CL system becomes unstable?
Modeling Model Uncertainty

- Uncertainty Models
  let \( G(z) \) be the actual system and \( G_0(z) \) be the model

- Additive uncertainty:
  \[
  G(z) = G_0(z) + \Delta_A(z) \quad \Rightarrow \quad \Delta_A(z) = G(z) - G_0(z)
  \]

- Multiplicative uncertainty:
  \[
  G(z) = [1 + \Delta_M(z)] \cdot G_0(z) \quad \Rightarrow \quad \Delta_M(z) = \frac{G(z) - G_0(z)}{G_0(z)} = \frac{\Delta_A(z)}{G_0(z)}
  \]

- Example:
  \[
  \forall \omega, \quad \left| \Delta_M(e^{j\omega T}) \right| < \delta_M(\omega)
  \]
Robust Stability

Let $L_0(z) = G_0(z)C(z)$ be the nominal loop transfer function and $L(z) = G(z)C(z)$ be the actual loop transfer function, then

$$\frac{1+L_0(z)}{1} > \text{Re}[L(z)] > \text{Im}[L(z)]$$

If

$$\|C(e^{j\omega T})\Delta_A(e^{j\omega T})\| < \|1 + L_0(e^{j\omega T})\| \quad \forall \omega T$$

then the actual feedback control pair $(G(z), C(z))$ is stable.

Q: Did we make any implicit assumption?
Robust Stability

- Robust Stability Theorem
  Consider a plant with a nominal plant model $G_0(z)$ and an actual pulse transfer function of $G(z)$. Assuming
  - the controller $C(z)$ stabilizes the nominal plant $G_0(z)$ and
  - the nominal loop transfer function $L_0(z) = G_0(z)C(z)$ and the actual loop transfer function $L(z) = G(z)C(z)$ have the same number of unstable poles

Then, a sufficient condition for the controller $C(z)$ to stabilize the actual plant is

$$
\left\| C(e^{j\omega T})\Delta_A(e^{j\omega T}) \right\| < \left\| 1 + L_0(e^{j\omega T}) \right\| \quad \forall \omega T
$$

or

$$
\left\| T_0(e^{j\omega T})\Delta_M(e^{j\omega T}) \right\| = \left\| \frac{L_0}{1 + L_0} \cdot \Delta_M(e^{j\omega T}) \right\| < 1 \quad \forall \omega T
$$
Sensitivity and Robustness

Observations:

- It is important to know the number of unstable poles (and zeros) of the plant.
- It is not critical to know the disturbances and the plant precisely for those frequencies of which the loop gain $||L(e^{j\omega T})||$ can be made large as long as CL stability is guaranteed.
- It is necessary to make the loop gain small for those frequencies for which the uncertainties in the plant, i.e., $||\Delta_M|| = ||\Delta_A/G_0||$, is large.
- The robust stability theorem is the LTI version of the small gain theorem.
- The modeling uncertainty $\Delta_M$ or $\Delta_A$ are usually unknown, in such a case, a known upper bound, $W_2(\omega)$, is used, i.e. the sufficient condition is replaced by $||T_0(e^{j\omega T})W_2(\omega)|| < 1$. 
Controllability and Observability

- **Controllability** characterizes the possibility of manipulating control inputs to steer the system (state variables) from any given initial state to any desired final state.

- **Observability** characterizes the possibility of determining (manifesting) the state variables of a dynamic system from observations of the inputs and the outputs.
**Controllability**

- **Definition:** Controllable
  
  A system is said to be *controllable* if it is possible to find a control sequence $u(k)$ to drive the system from any initial state to any arbitrary finite destination state in finite time.

Given an LTI system

$$
\begin{align*}
    x(k + 1) &= A \cdot x(k) + B \cdot u(k) \\
    y(k) &= C \cdot x(k) + D \cdot u(k)
\end{align*}
$$

$x(k)$: $n \times 1$ state vector

An $n$-th order LTI system (shown above) is *controllable* if and only if the following *Controllability Matrix* $W_C$ has rank $n$:

$$
W_C = \begin{bmatrix}
    B & AB & A^2B & \cdots & A^{n-1}B
\end{bmatrix}
$$

– *since controllability depends only on the matrices $A$ and $B$, we often say $(A, B)$ is controllable.*
Controllability

Proof:
Iterative solution of LTI system at time index $n$

$$x(n) = A^n x(0) + \sum_{j=0}^{n-1} A^{n-j-1} B \cdot u(j)$$

$$\Rightarrow x(n) - A^n x(0) = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u(n-1) \\ u(n-2) \\ \vdots \\ u(0) \end{bmatrix} = W_C \cdot U$$

Given any initial state $x(0)$ and any final state $x(n)$, it is possible to find the required control sequence $U$ if and only if the columns of $W_C$ spans the entire $\mathbb{R}^n$ space.
Controllability

Observations:

■ Matrix $W_C$ has a dimension $n \times nm$, where $m$ is the number of input signals. If $m = 1$ (single input), $W_C$ is a square matrix. Then the condition of having rank $n$ is equivalent to requiring

$$\det(W_C) \neq 0$$

■ Including more than $n$ control input sequence, i.e., using longer time, will not change the reachable space of the system. From the Cayley-Hamilton Theorem, $A^nB$ can be expressed as a linear combination of $B$, $AB$, ... $A^{n-1}B$. Therefore, $A^nB$ will not introduce additional linearly independent vectors.

⇒ any destination vector in the state space should be reachable in $n$ sample periods.
Controllability

Example: Given the following continuous-time system

\[
\begin{bmatrix}
    \frac{dx_1}{dt} \\
    \frac{dx_2}{dt}
\end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot u
\]

The ZOH equivalent D.T. system for \( T = 1 \) can be written as:

\[
\begin{align*}
    x_1(k+1) &= 0.368 x_1(k) + 0.632 x_2(k) \\
    x_2(k+1) &= 0.135 x_2(k)
\end{align*}
\]

\[
\Rightarrow W_c = \begin{bmatrix} 0.632 & 0.368 \cdot 0.632 \\ 0 & 0 \end{bmatrix}
\]

\Rightarrow \text{Not controllable} (obviously)
Controllability

**Example:** Given two LTI systems:

System 1:
\[
\begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x(k)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
u(k) \quad \text{det}(W_C) = \text{det}\left(\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}\right) = -1
\]

System 2:
\[
\begin{bmatrix}
0 & 0.01 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
x(k) \\
x(k)
\end{bmatrix}
+ 
\begin{bmatrix}
0 \\
1
\end{bmatrix}
u(k) \quad \text{det}(W_C) = \text{det}\left(\begin{bmatrix}
0 & 0.01 \\
1 & 1
\end{bmatrix}\right)
= -0.01
\]

To drive the systems from initial state \((0,0)\) to a final state of \((1, 2)\) in two steps, the control signal \([u(0), u(1)]\) will be

\[
\begin{bmatrix}
1, & 2
\end{bmatrix} \quad \text{for system 1,}
\]

\[
\begin{bmatrix}
100, & -98
\end{bmatrix} \quad \text{for system 2.}
\]
Controllable Canonical Form

\[
\begin{bmatrix}
  x_1(k+1) \\
  x_2(k+1) \\
  \vdots \\
  x_{n-1}(k+1) \\
  x_n(k+1)
\end{bmatrix}
= \begin{bmatrix}
  0 & 1 & 0 & \cdots & 0 & 0 \\
  0 & 0 & 1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & 0 & 1 \\
  -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1
\end{bmatrix}
\begin{bmatrix}
  x_1(k) \\
  x_2(k) \\
  \vdots \\
  x_{n-1}(k) \\
  x_n(k)
\end{bmatrix}
+ \begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0 \\
  1
\end{bmatrix}
\cdot u(k)
\]

\[
y(k) = \begin{bmatrix}
  b_n & b_{n-1} & \cdots & b_1
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\]

\[
W_C = \begin{bmatrix}
  B & AB & \cdots & A^n B
\end{bmatrix}
= \begin{bmatrix}
  0 & 0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & \vdots & 1 & -a_1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 1 & -a_1 & \vdots & \vdots \\
  0 & 1 & -a_1 & a_1^2 - a_2 & \vdots & \vdots \\
  1 & -a_1 & a_1^2 - a_2 & -a_1^3 + 2a_1a_2 - a_3 & \cdots & \cdots
\end{bmatrix}
\]

\[\Rightarrow \quad \det(W_C) = 1 \text{ (controllable)}\]
Controllability

Controllability is a system property and is independent of the choice of states (related by similarity transformation).

Let \( x \) be the state vector that is associated with the system representation \([A \ B \ C \ D]\) and that \( x_{\text{NEW}} = T x \) is associated with the representation \([A_{\text{NEW}} \ B_{\text{NEW}} \ C_{\text{NEW}} \ D]\). If \((A, B)\) is controllable, i.e.

\[
\text{Rank}(W_C) = \text{Rank} \begin{bmatrix} B & AB & L & A^{n-1}B \end{bmatrix} = n
\]

\[\Rightarrow W_{\text{CNEW}} = \begin{bmatrix} B_{\text{NEW}} & A_{\text{NEW}}B_{\text{NEW}} & L & A_{\text{NEW}}^{n-1}B_{\text{NEW}} \end{bmatrix} = \begin{bmatrix} TB & TAT^{-1}TB & L & TA^{n-1}T^{-1}TB \end{bmatrix} = T \cdot W_C\]

Since \( T \) is non-singular,

\[\Rightarrow \text{Rank}(W_{\text{CNEW}}) = n\]

If one SS representation is controllable, all SS representation related by similarity transformations are also controllable.
Observability

Definition: Observable

A system is said to be observable if the initial state \( x(0) \) can be reconstructed from observing the output \( y(k) \) in finite sample steps.

Given an LTI system

\[
\begin{align*}
    x(k+1) &= A \cdot x(k) + B \cdot u(k) \\
    y(k) &= C \cdot x(k) + D \cdot u(k)
\end{align*}
\]

An \( n \)th order LTI system (shown above) is observable if and only if the following Observability Matrix \( W_o \) has rank \( n \):

\[
W_c = \begin{bmatrix} \text{C} \\ \text{CA} \\ \text{M} \\ \text{CA}^{n-1} \end{bmatrix}
\]

– since observability depends only on the matrices \( A \) and \( C \), we often say \((A, C)\) is observable.
Observability

**Proof:**
Without losing generality, we can assume that \( u(k) = 0 \) during the entire observation period, then the observation matrix \( Y \) can be written as

\[
Y = \begin{bmatrix}
y(0) \\
y(1) \\
\vdots \\
y(n-1)
\end{bmatrix} = \begin{bmatrix}
C \\
CA \\
\vdots \\
CA^{n-1}
\end{bmatrix} \cdot x(0) = W_o \cdot x(0)
\]

When \( W_o \) has rank \( n \), we can select \( n \) independent rows of \( W_o \) and reconstruct \( x(0) \) simply by using the matrix inversion and the corresponding output measurements \( Y \), or

\[
x(0) = \left( W_o^T W_o \right)^{-1} \cdot W_o^T Y
\]
Observability

Observations:

- Matrix $W_O$ has a dimension $nm \times n$, where $m$ is the number of output signals. If $m = 1$ (single output), $W_O$ is a square matrix. Then the condition of having rank $n$ is equivalent to requiring
  $$\det(W_O) \neq 0$$

- Including more than $n$ output measurements, i.e. using longer time, will not change the observable space of the system. From the Cayley-Hamilton Theorem, $CA^n$ can be expressed as a linear combination of $C, CA, ..., CA^{n-1}$. Therefore, $CA^n$ will not introduce additional linearly independent vectors.

$$\Rightarrow \text{any initial vector in the state space should be detected (observed) within } n \text{ sample periods.}$$
Observable Canonical Form

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1) \\
    \vdots \\
    x_{n-1}(k+1) \\
    x_n(k+1)
\end{bmatrix}
= \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 & -a_n \\
    1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & 1 & 0 & -a_2 \\
    0 & 0 & \cdots & 0 & 1 & -a_1
\end{bmatrix}
\begin{bmatrix}
    x_1(k) \\
    x_2(k) \\
    \vdots \\
    x_{n-1}(k) \\
    x_n(k)
\end{bmatrix}
+ \begin{bmatrix}
    b_n \\
    b_{n-1} \\
    \vdots \\
    b_2 \\
    b_1
\end{bmatrix}
\cdot u(k)
\]

\[
y(k) = \begin{bmatrix}
    0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\cdot \begin{bmatrix}
    x_1(k) \\
    x_2(k) \\
    \vdots \\
    x_{n-1}(k) \\
    x_n(k)
\end{bmatrix}
\]

\[
W_0 = \begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{n-1}
\end{bmatrix}
= \begin{bmatrix}
    0 & 0 & \cdots & 0 & 0 & 1 \\
    0 & 0 & \cdots & 0 & 1 & -a_1 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & 0 & \cdots & -a_1 & a_1^2 - a_2 & -a_1^3 + 2a_1a_2 - a_3 \\
    0 & 1 & \cdots & a_1^2 - a_2 & \vdots & \vdots \\
    1 & -a_1 & \cdots & \vdots & \vdots & \vdots
\end{bmatrix}
\]

\[
\Rightarrow \det(W_0) = 1 \text{ (observable)}
\]
Controllability and Observability

- **Principle of Duality**
  Controllability and observability are dual properties (characteristics) of a system.
  Consider a system $S_1$ represented by $[A, B, C, D]$ and its *dual* (transposed) system $S_2$ represented by $[A^T, C^T, B^T, D^T]$, then

- The system $S_1$ is controllable $\iff$ the system $S_2$ is observable
- The system $S_1$ is observable $\iff$ the system $S_2$ is controllable
Effect of Sampling (Discretization)

If a continuous-time system is both controllable and observable, then the ZOH equivalent discrete-time system is both controllable and observable if and only if, for every eigenvalue of the continuous-time system, the relationship

$$\text{Re}(\lambda_i) = \text{Re}(\lambda_j)$$

implies

$$\text{Im}(\lambda_i - \lambda_j) \neq \frac{2n\pi}{T}$$

where $T$ is the sampling period and $n = \pm 1, \pm 2, \ldots$

If the continuous-time system does not have any complex poles, controllability and observability are preserved for the corresponding ZOH equivalent discrete-time system.
Effect of Sampling (Discretization)

Example: Given a continuous-time system

\[ G(s) = \frac{\omega^2}{s^2 + \omega^2} \quad \text{or} \quad \dot{x}(t) = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \omega \end{bmatrix} u(t) \]

\[ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \]

The above C.T. system is both controllable and observable!

The corresponding ZOH equivalent discrete-time system is

\[ x(k+1) = \begin{bmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{bmatrix} x(k) + \begin{bmatrix} 1 - \cos(\omega T) \\ \sin(\omega T) \end{bmatrix} u(k) \]

\[ y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \]

\[ \Rightarrow \quad \det(W_c) = -2 \sin(\omega T) \cdot (1 - \cos(\omega T)) \]

\[ \quad \det(W_o) = \sin(\omega T) \]

Controllability and observability is preserved if \( T \neq n\pi/\omega \)
Pole-Zero Cancellation

- Loss of controllability and observability through pole/zero cancellation

An $n$-th order system

$$x(k + 1) = A \cdot x(k) + B \cdot u(k)$$

$$y(k) = C \cdot x(k) + D \cdot u(k)$$

is both controllable and observable if and only if the corresponding transfer function

$$G(z) = C \left(z I - A\right)^{-1} B + D$$

is an $n^{th}$ order rational function without any pole/zero cancellation.

If pole/zero cancellation occurs, then the system is either uncontrollable, or unobservable, or both.
Pole-Zero Cancellation

Example: Pole/zero cancellation and loss of observability
Consider a 3\textsuperscript{rd} order system in controllable canonical form

\[
x(k+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_2 & -a_1 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(k) \implies G(z) = \frac{b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z}
\]

\[
y(k) = \begin{bmatrix} b_3 & b_2 & 0 \end{bmatrix} x(k)
\]

\[
W_o = \begin{bmatrix} b_3 & b_2 & 0 \\ 0 & b_3 & b_2 \\ 0 & -a_2 b_2 & b_3 - a_1 b_2 \end{bmatrix} \implies \det(W_o) = b_3^3 - a_1 b_2 b_3^2 + a_2 b_2^2 b_3
\]

For observability, the determinant of the observability matrix must be non-zero.

\[
The system will be unobservable (\det(W_o) = 0), if the zero of the transfer function (z = -b_3/b_2) is also a root of the characteristic equation (z^3 + a_1 z^2 + a_2 z), i.e., a pole of the transfer function, leading to pole/zero cancellation in G(z).
\]
**Pole-Zero Cancellation**

**Example:** Given a 2nd order system in controllable canonical form:

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1) \\
    u(k+1)
\end{bmatrix} =
\begin{bmatrix}
    -0.7 & -0.1 & 1 \\
    1 & 0 & 0 \\
    0 & 0 & -0.8
\end{bmatrix}
\begin{bmatrix}
    x_1(k) \\
    x_2(k) \\
    u(k)
\end{bmatrix}
\begin{bmatrix}
    0
\end{bmatrix}
\]

\[
y(k) = \begin{bmatrix}
    1 & 0.8
\end{bmatrix}
\begin{bmatrix}
    x_1(k) \\
    x_2(k)
\end{bmatrix}
\]

\[\Rightarrow G(z) = \frac{z + 0.8}{(z + 0.2)(z + 0.5)}\]

A direct pole/zero cancellation controller is used:

The combined system can be written as

\[
\begin{bmatrix}
    x_1(k+1) \\
    x_2(k+1) \\
    v(k)
\end{bmatrix} =
\begin{bmatrix}
    -0.7 & -0.1 & 1 \\
    1 & 0 & 0 \\
    0 & 0 & -0.8
\end{bmatrix}
\begin{bmatrix}
    x_1(k) \\
    x_2(k) \\
    v(k)
\end{bmatrix}
\begin{bmatrix}
    0
\end{bmatrix}
\]

\[y(k) = \begin{bmatrix}
    1 & 0.8 & 0
\end{bmatrix} x(k)\]

The combined system is controllable but not observable!