The Continuous-Time System and Its ZOH Equivalent Discrete-Time System

Continuous-Time

ZOH Equivalent Discrete-Time

$$\left[\begin{array}{c|c} \mathbf{F} & \mathbf{G} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

$$\mathbf{A} = e^{\mathbf{F}T}, \quad \mathbf{B} = \left[\int_0^T e^{\mathbf{F}\eta}\, d\eta\right]\mathbf{G}$$

$\Rightarrow p_{si}$ is an eigenvalue of F

- The poles of the continuous-time system $\{p_{si}\}$ and the poles of the ZOH equivalent discrete-time system $\{p_{zi}\}$ satisfies the following relationship:

$$p_{zi} = e^{p_{si}T}$$

$\Rightarrow \exists\, v_i \neq 0$ s.t. $F v_i = p_{si} v_i$

$\Rightarrow F^2 v_i = F(F v_i) = F(p_{si} v_i) = p_{si} F v_i = p_{si}^2 v_i$

- The eigenvectors of the ZOH equivalent discrete-time system are the same as those of the original continuous-time system

$\Rightarrow \forall k,\ F^k v_i = p_{si}^k v_i$

$\Rightarrow e^{FT} v_i = e^{p_{si}T} v_i \Rightarrow A v_i = e^{p_{si}T} v_i$

The Continuous-Time System and Its ZOH Equivalent Discrete-Time System

Two linear algebra tools are needed to prove the previous claim:

- **The Cayley-Hamilton Theorem**

  A square matrix $\mathbf{A}$ satisfies its own characteristic equation

  - Any function of a square matrix (any matrix function) can be represented by a polynomial function of this matrix with degree less than n (where n is the size of the matrix).

- **Function of a Matrix**

  If $f(\mathbf{A})$ is a finite-order polynomial of matrix $\mathbf{A}$, and $\mathbf{e}_i$ are the eigenvectors of the matrix $\mathbf{A}$ associated with the eigenvalues λ_i then $f(\lambda_i)$ are eigenvalues of $f(\mathbf{A})$ associated with eigenvectors $\mathbf{e}_i$, i.e.

$$f(\mathbf{A}) \cdot \mathbf{e}_i = f(\lambda_i) \cdot \mathbf{e}_i$$

The Cayley-Hamilton Theorem

A square matrix **A** satisfies its own characteristic equation:

$$A(s) = \det\left(s\mathbf{I} - \mathbf{A}\right) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$$

$$= (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_n) = 0$$

$$\Rightarrow \quad A(\mathbf{A}) = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} = 0$$

Proof:

Note that $(s\mathbf{I} - \mathbf{A})^{-1}$ can be represented by

$$\left(s\mathbf{I} - \mathbf{A}\right)^{-1} = \frac{1}{A(s)}\text{Adj}(s\mathbf{I} - \mathbf{A}) = \frac{\mathbf{B}_1 s^{n-1} + \mathbf{B}_2 s^{n-2} + \cdots + \mathbf{B}_{n-1}s + \mathbf{B}_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$$

$$\Rightarrow \quad \left(\mathbf{B}_1 s^{n-1} + \mathbf{B}_2 s^{n-2} + \cdots + \mathbf{B}_{n-1}s + \mathbf{B}_n\right)\left(s\mathbf{I} - \mathbf{A}\right) = \left(s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n\right)\cdot\mathbf{I}$$

Compare coefficients:

$$\mathbf{B}_1 = \mathbf{I} \qquad \mathbf{B}_2 = \mathbf{A}\mathbf{B}_1 + a_1\mathbf{I} \;\Rightarrow\; \mathbf{B}_2 = \mathbf{A} + a_1\mathbf{I}$$

$$\mathbf{B}_3 = \mathbf{A}\mathbf{B}_2 + a_2\mathbf{I} \;\Rightarrow\; \mathbf{B}_3 = \mathbf{A}^2 + a_1\mathbf{A} + a_2\mathbf{I}$$

$$\mathbf{B}_{n-1} = \mathbf{A}\mathbf{B}_{n-2} + a_{n-2}\mathbf{I} \;\Rightarrow\; \mathbf{B}_{n-1} = \mathbf{A}^{n-2} + a_1\mathbf{A}^{n-3} + \cdots + a_{n-3}\mathbf{A} + a_{n-2}\mathbf{I}$$

$$\mathbf{B}_n = \mathbf{A}\mathbf{B}_{n-1} + a_{n-1}\mathbf{I} \Rightarrow \mathbf{B}_n = \mathbf{A}^{n-1} + a_1\mathbf{A}^{n-2} + \cdots + a_{n-2}\mathbf{A} + a_{n-1}\mathbf{I}$$

$$\mathbf{0} = \mathbf{A}\mathbf{B}_n + a_n\mathbf{I} \Rightarrow \mathbf{0} = \mathbf{A}^n + a_1\mathbf{A}^{n-1} + \cdots + a_{n-1}\mathbf{A} + a_n\mathbf{I} \quad \Rightarrow \quad A(\mathbf{A}) = \mathbf{0}$$

Function of a Matrix

If $f(\mathbf{A})$ is a finite-degree polynomial of matrix $\mathbf{A}$, and $\mathbf{e}_i$ are the eigenvectors of the matrix $\mathbf{A}$ associated with the eigenvalues λ_i , if then

$$f(\mathbf{A})\cdot\mathbf{e}_i = f(\lambda_i)\cdot\mathbf{e}_i$$

In other words, $f(\lambda_i)$ are the eigenvalues of the matrix $f(\mathbf{A})$, and that the eigenvectors $\mathbf{e}_i$ will be unchanged.

Proof:

Let $(\lambda_i, \mathbf{e}_i)$ be the eigen-pairs for matrix $\mathbf{A}$ then

$$\mathbf{A}\cdot\mathbf{e}_i = \lambda_i\cdot\mathbf{e}_i \quad \Rightarrow \quad \mathbf{A}^2\cdot\mathbf{e}_i = \lambda_i\cdot\mathbf{A}\cdot\mathbf{e}_i = \lambda_i^2\cdot\mathbf{e}_i \quad \Rightarrow\cdots\Rightarrow \quad \mathbf{A}^n\cdot\mathbf{e}_i = \lambda_i^n\cdot\mathbf{e}_i$$

Suppose $f(\mathbf{A})$ can be written as

$$f(\mathbf{A}) = c_0\mathbf{A}^n + c_1\mathbf{A}^{n-1} + \cdots + c_{n-1}\mathbf{A} + c_n\mathbf{I}$$

Then

$$\begin{aligned} f(\mathbf{A})\cdot\mathbf{e}_i &= \left(c_0\mathbf{A}^n + c_1\mathbf{A}^{n-1} + \cdots + c_{n-1}\mathbf{A} + c_n\mathbf{I}\right)\cdot\mathbf{e}_i \\ &= \left(c_0\lambda_i^n + c_1\lambda_i^{n-1} + \cdots + c_{n-1}\lambda_i + c_n\right)\cdot\mathbf{e}_i = f(\lambda_i)\cdot\mathbf{e}_i \end{aligned}$$

$$\Rightarrow \quad f(\mathbf{A})\cdot\mathbf{e}_i = f(\lambda_i)\cdot\mathbf{e}_i$$

The Continuous-Time System and Its ZOH Equivalent Discrete-Time System

- The poles of the continuous-time system $\{p_{si}\}$ and the poles of the ZOH equivalent discrete-time system $\{p_{zi}\}$ satisfy the following relationship:

$$p_{zi} = e^{p_{si}T}$$

Proof:

Let $f(\mathbf{F}) = e^{\mathbf{F}T}$

Then $\mathbf{A} = e^{\mathbf{F}T} = f(\mathbf{F})$

Thus $\forall \{p_{si}, \mathbf{e}_i\}$ with $\mathbf{F} \cdot \mathbf{e}_i = p_{si}\mathbf{e}_i$:

$$\Rightarrow \quad \mathbf{A} \cdot \mathbf{e}_i = f(\mathbf{F}) \cdot \mathbf{e}_i = f(p_{si}) \cdot \mathbf{e}_i = e^{p_{si}T} \cdot \mathbf{e}_i$$

$$\Rightarrow \quad \left(f(p_{si}) \cdot \mathbf{I} - \mathbf{A} \right) \cdot \mathbf{e}_i = 0$$

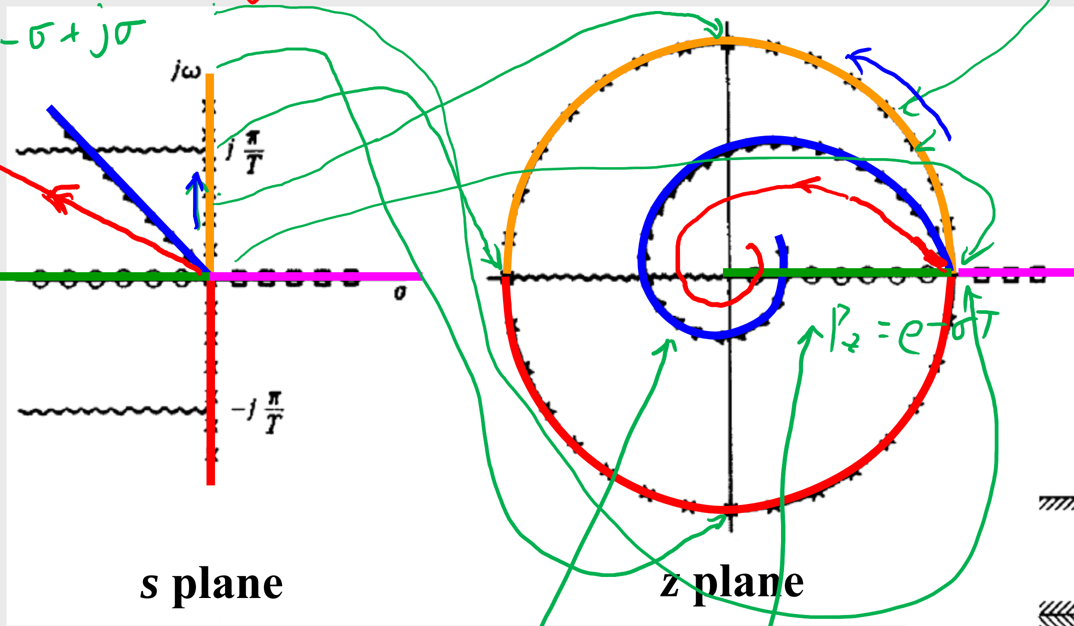
$\Rightarrow \quad f(p_{si})$ is the eigenvalue of $\mathbf{A}$ with eigenvector of $\mathbf{e}_i$, i.e.

$$p_{zi} = f(p_{si}) = e^{p_{si}T}$$

$\Rightarrow$ **Note that eigenvectors of the C.T. and the D.T. systems are the same !**

Mapping Between C.T. and D.T. Poles

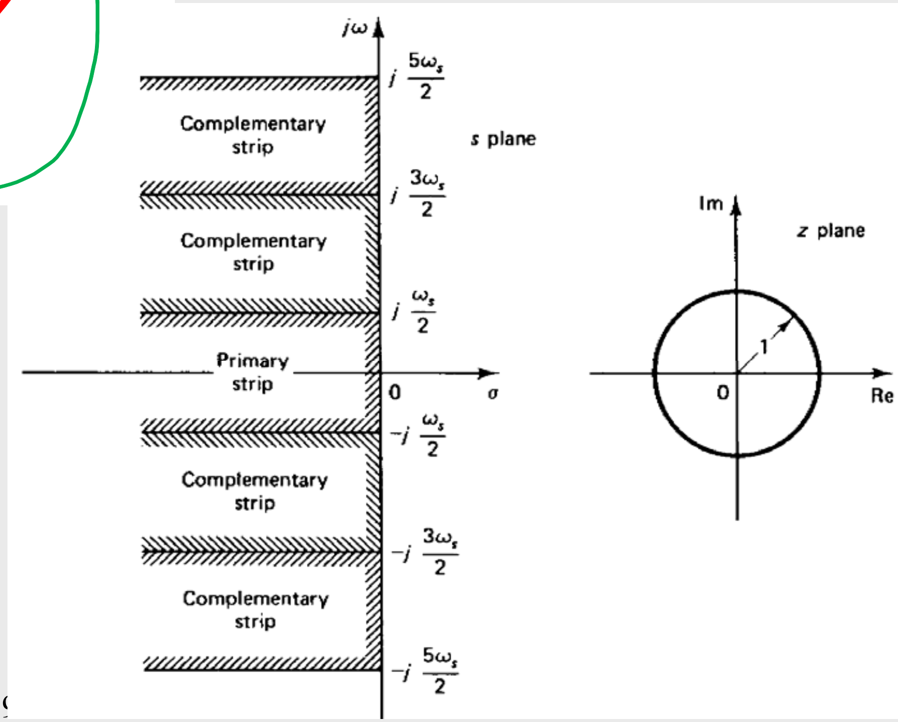
$p_s = j\omega \Rightarrow p_z = e^{p_s T} = e^{j\omega T} \Rightarrow |p_z| = 1$ & $\angle p_z = \omega T = \begin{cases} 0 & \omega = 0 \\ \pi & \omega = \omega_N = \frac{\pi}{T} \end{cases}$



$p_z = e^{-\sigma(1+j)T} = e^{-\sigma T} \cdot e^{-\sigma T j}$

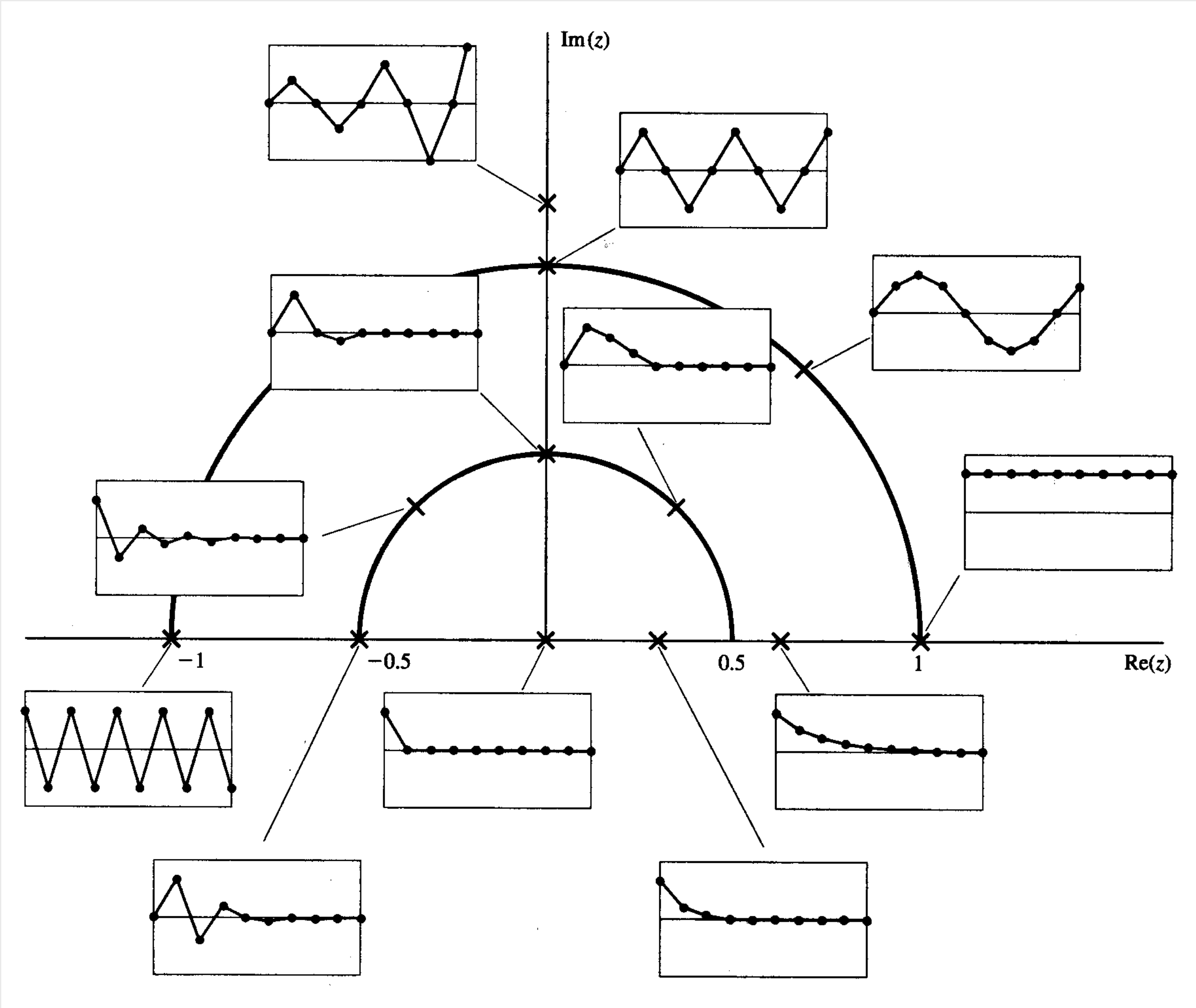
$\Rightarrow |p_z| = e^{-\sigma T} \xrightarrow{\sigma \to \infty} 0$

$\angle p_z = -\sigma T \xrightarrow{\sigma \to \infty} -\infty$



— K. Ogata, *Discrete-Time Control Systems*, 2nd Ed., Prentice-Hall, 19

D.T. Free Response and Pole Location



— G.F. Franklin, J.D. Powell, M.Workman, *Digital Control of Dynamic Systems*, 3rd Ed., Addison-Wesley, 1998

C.T. and D.T. System Zeros

Do the *zeros of the continuous-time system* $\{z_{si}\}$ and the *zeros of the discrete-time system* $\{z_{zi}\}$ satisfy similar relationship?

$$z_{zi} \overset{?}{=} e^{z_{si}T}$$

No!

Unlike poles, zeros of the ZOH equivalent DT system and those of the CT system do not have a clear relationship. In general, sampled data model has a relative degree of 1, regardless of the relative degree of the CT system, i.e., DT system has *n-1* zeros regardless of *m* of its CT counter. The extra zeros of DT system are referred to as the sampling zeros. For fast sampling, those *m* zeros that can be thought of as being mappings of continuous time zeros are approximated through the relationship ($z_{zi} = e^{z_{si}T}$). The remaining *n-m-1=r-1* sampling zeros will converge to finite zeros in the left half plane determined by $B_r(z) = 0$ where

$$B_1(z) = 1, \quad B_2(z) = z + 1$$

$$B_3(z) = (z-1)^2 + 6(z-1) + 6$$

$= z^2 - 2z + 1 + 6z - 6 + 6$
$= z^2 + 4z + 1$ $\Rightarrow z_{1,2} = \frac{-4 \pm \sqrt{12}}{2} = -3.7$ & -0.26

$$B_4(z) = (z-1)^3 + 14(z-1)^2 + 36(z-1) + 24$$

$$\vdots$$

$\Rightarrow$ Unstable zeros
or
NMP DT system

for relative degree more than 2 CT system
$r > 2$

C.T. and D.T. System Zeros

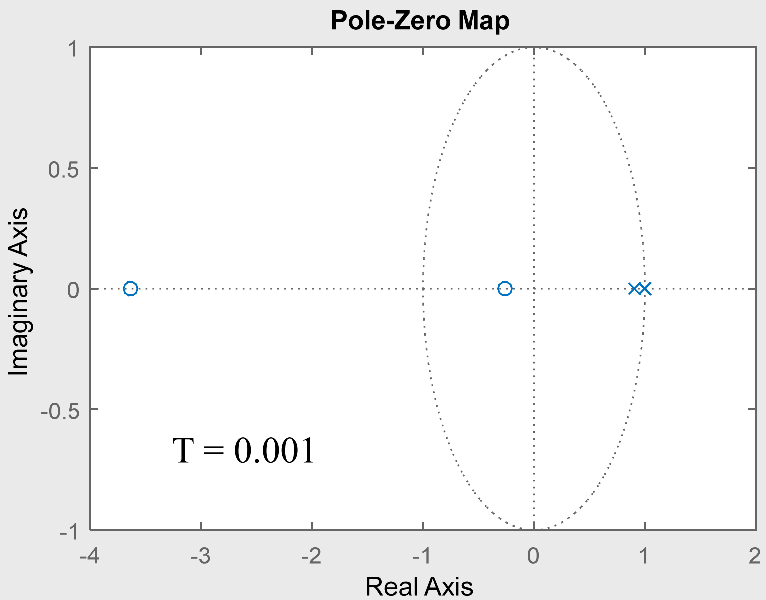
Example:

$$G_p(s) = \frac{500}{s(s+5)(s+100)} \quad \Rightarrow \quad P(z) = \frac{b_0 z^2 + b_1 z + b_2}{(z-1)(z-e^{-5T})(z-e^{-100T})}$$

$$b_0 = T - (399 - 400e^{-5T} + e^{-100T})/1900$$

$$b_1 = \left(399 - (401 + 1900T)e^{-5T} + (401 - 1900T)e^{-100T} - 399e^{-105T}\right)/1900$$

$$b_2 = \left(e^{-5T} - 400e^{-100T} + (399 + 1900Te^{-105T}\right)/1900$$



P =

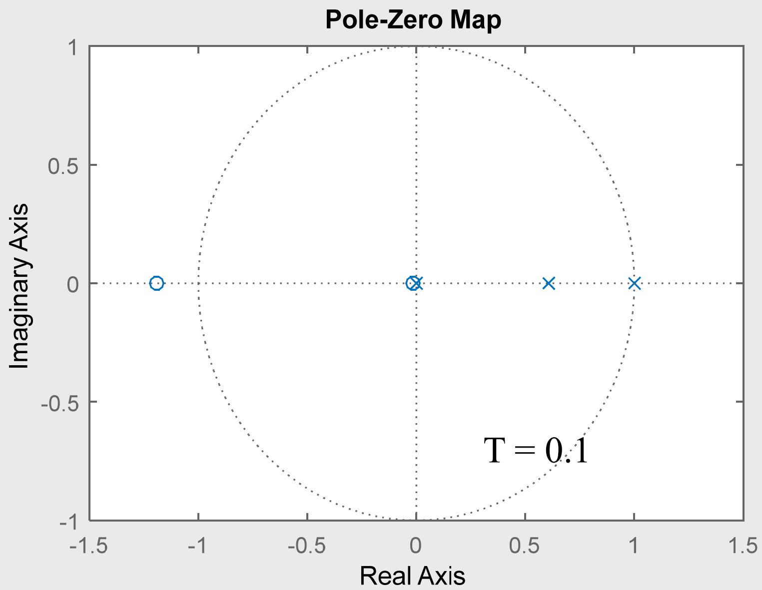
8.119e-08 z^2 + 3.164e-07 z + 7.704e-08

----

z^3 - 2.9 z^2 + 2.8 z - 0.9003

z = -3.6361, -0.2610

p = 1.0000, 0.9950, 0.9048



P =

0.01769 z^2 + 0.02134 z + 0.0003182

----

z^3 - 1.607 z^2 + 0.6066 z - 2.754e-05

z = -1.1910, -0.0151

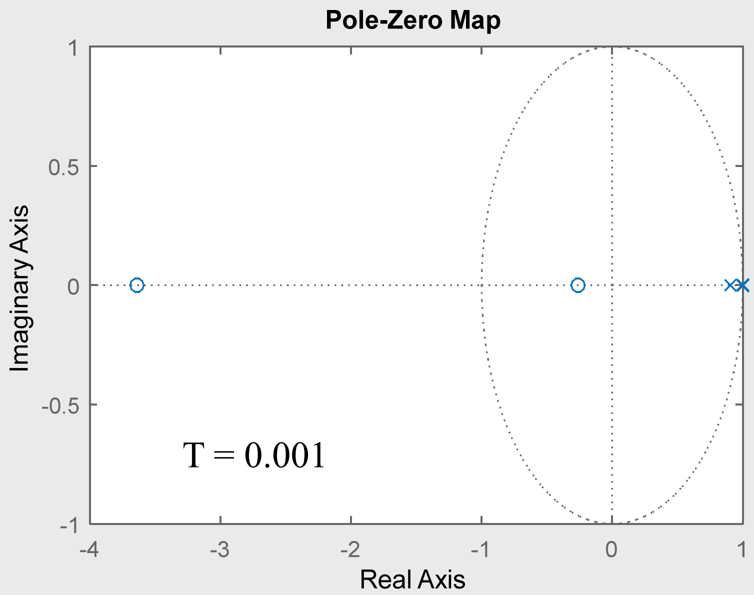
p = 1.0000, 0.6065, 0.0000

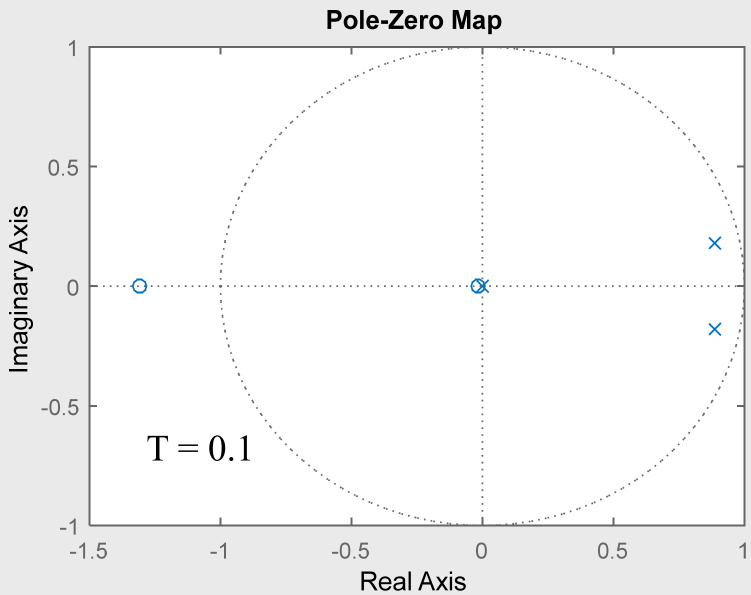
C.T. and D.T. System Zeros

Example:

$$G_p(s) = \frac{500}{(s^2+2s+5)(s+100)} \quad \Rightarrow \quad P(z) = \frac{b_0 z^2 + b_1 z + b_2}{(z-e^{(-1+j2)T})(z-e^{(-1-j2)T})(z-e^{-100T})}$$

$b_0 = \ldots$
$b_1 = \ldots$
$b_2 = \ldots$






```
P =
  8.125e-08 z^2 + 3.169e-07 z + 7.721e-08
 ---------------------------------------
    z^3 - 2.903 z^2 + 2.806 z - 0.903

z = -3.6389,   -0.2611
p = 0.9990 ± 0.0020i,   0.9048
```

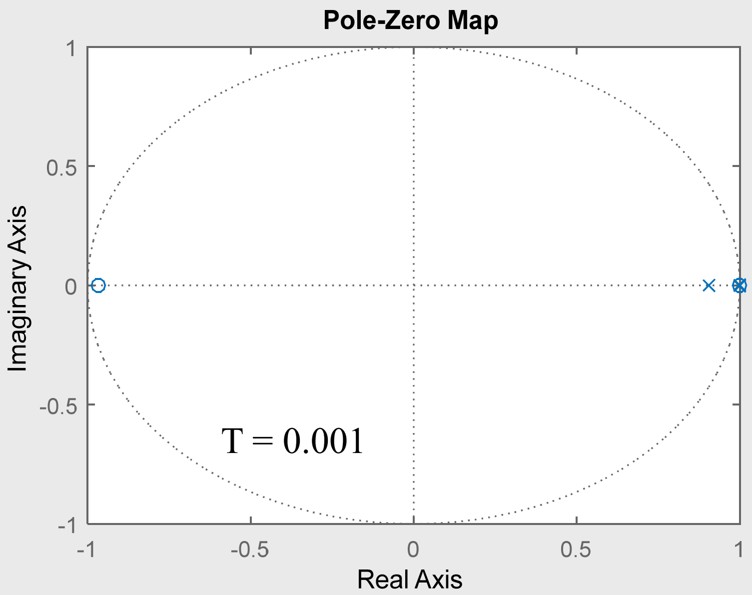
```
P =
 0.01923 z^2 + 0.02548 z + 0.0004163
 -------------------------------------
 z^3 - 1.774 z^2 + 0.8188 z - 3.717e-05

z = -1.3083,    -0.0165
p =  0.8868 ± 0.1798i,  0.0000 + 0.0000i
```

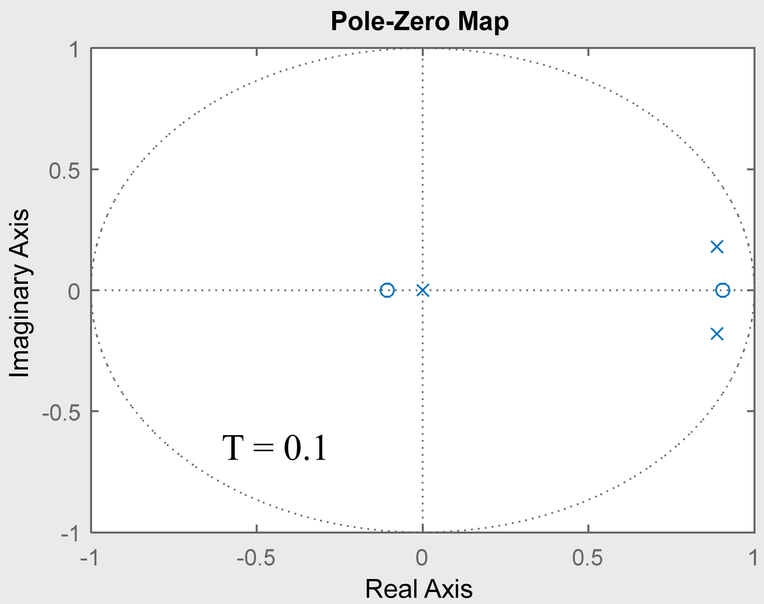
C.T. and D.T. System Zeros

Example:

$$G_p(s)=\frac{500(s+1)}{(s^2+2s+5)(s+100)} \Rightarrow P(z)=\frac{b_0z^2+b_1z+b_2}{(z-e^{(-1+j2)T})(z-e^{(-1-j2)T})(z-e^{-100T})}$$

$$b_0=\ldots$$
$$b_1=\ldots$$
$$b_2=\ldots$$





P =

0.0002418 z^2 - 7.763e-06 z - 0.0002336
----
z^3 - 2.903 z^2 + 2.806 z - 0.903

z = 0.9990, -0.9669

p = 0.9990 ± 0.0020i, 0.9048 + 0.0000i

P =

0.4278 z^2 - 0.3413 z - 0.04131
----
z^3 - 1.774 z^2 + 0.8188 z - 3.717e-05

z = 0.9047, -0.1067

p = 0.8868 ± 0.1798i, 0.0000 + 0.0000i

What are Zeros Anyway?

- z_{zi} is said to be the zero of a pulse transfer function $G(z)$, if at $z = z_{zi}$, $G(z_{zi}) = \mathbf{C}(z_{zi}\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = 0$
- Given a minimum (i.e., controllable and observable) DT state space representation of a square system (number of inputs = number of outputs) , zeros of the system are defined as the z_{zi}'s that satisfy

$$\det\left[\begin{array}{c|c} z_{zi}\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = 0$$

The above equation implies that

$$\exists \;\; non\text{-}zero \text{ pair } (\mathbf{e}_i, u) \text{ that satisfies } \begin{bmatrix} z_{zi}\mathbf{I} - \mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{e}_i \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ ***At the system zeros, there exists non-zero inputs and states such that the system outputs are identically zero.***

System Zeros

- To prove the above, we need to know the following two facts in linear algebra:
  - **Fact I:**

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

  - **Fact II:** Given two square matrices $\mathbf{M}$ and $\mathbf{N}$, then

$$\det(\mathbf{M} \cdot \mathbf{N}) = \det(\mathbf{M}) \cdot \det(\mathbf{N})$$

- **Proof:**

Use Fact I:

$$\begin{bmatrix} z\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} z\mathbf{I}-\mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -(z\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

Use Fact II:

$$\det\begin{bmatrix} z\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det\left(z\mathbf{I}-\mathbf{A}\right) \cdot \det\left[\mathbf{D}+\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}\right]$$

Since $\det(z_{zi}\mathbf{I} - \mathbf{A}) \neq 0$ (why?), then z_{zi} is a zero of the system iff

$$\det\left[\begin{array}{c|c} z_{zi}\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] = 0$$

Example of Transmission Zeros

- **Example:**

$$G(z) = \frac{z-0.5}{(z-1)(z-0.8)} = \frac{z-0.5}{z^2-1.8z+0.8}$$

Convert to controllable canonical form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.8 & 1.8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k), \quad y(k) = \begin{bmatrix} -0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 0 \cdot u(k)$$

System has a zero at $z = 0.5$, i.e.

$$\begin{bmatrix} z\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0.5\mathbf{I}-\mathbf{A} & -\mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0.5 & -1 & 0 \\ 0.8 & -1.3 & -1 \\ -0.5 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_2(k) = 0.5x_1(k) \\ u(k) = 0.15x_1(k) \end{cases}$$

	$k=0$	$k=1$	$k=2$	$k=3$
$x_1(k)$	1	0.5	0.25	0.125
$x_2(k)$	0.5	0.25	0.125	0.0625
$u(k)$	0.15	0.075	0.0375	0.01875
$y(k)$	0	0	0	0