State Space Representation

Definition: State

The state of a dynamic system is the set of smallest number independent variables (known as the *state variables*) such that the knowledge of these variables at time $t = t_0$, together with the knowledge of the input for $t \;,\; t_0$, completely determines the behavior of the system for any time $t \;,\; t_0$.

Continuous-Time

$$\left\{ \mathbf{x}(t) \in R^n , t \in R \right\}$$

Discrete-Time

$$\left\{ \mathbf{x}(k) \in R^n , k \in Z \right\}$$

Definition: State equations

The set of first order differential equations or difference equations that express the evolution of the state variables with respect to input vector.

Continuous-Time

$$\dot{\mathbf{x}}(t) = \mathbf{f}\left(\mathbf{x}(t), \mathbf{u}(t), t \right)$$

Discrete-Time

$$\mathbf{x}(k+1) = \mathbf{f}\left(\mathbf{x}(k), \mathbf{u}(k), k \right)$$

State Space Representation

- **Notations (LTI systems)**

Continuous-Time

$$\dot{\mathbf{x}}(t) = \mathbf{F}\cdot\mathbf{x}(t) + \mathbf{G}\cdot\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t)$$

Discrete-Time

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

$\mathbf{x}(t)$ or $\mathbf{x}(k)$	$n \times 1$ vector	State Vector
$\mathbf{u}(t)$ or $\mathbf{u}(k)$	$r \times 1$ vector	Input Vector
$\mathbf{y}(t)$ or $\mathbf{y}(k)$	$m \times 1$ vector	Output Vector
F or **A**	$n \times n$ matrix	System Matrix
G or **B**	$n \times r$ matrix	Input Matrix
C	$m \times n$ matrix	Output Matrix
D	$m \times r$ matrix	Direct Transmission Matrix

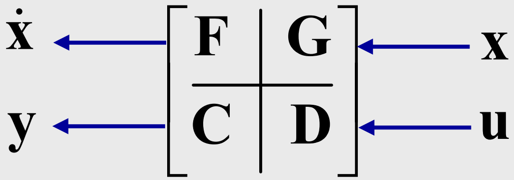
State Space Representation

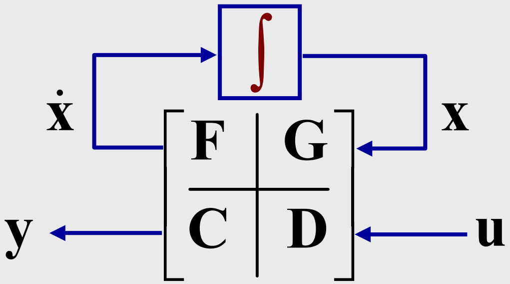
- Several Representations of LTI Systems:

Continuous-Time

$$\dot{\mathbf{x}}(t) = \mathbf{F}\cdot\mathbf{x}(t) + \mathbf{G}\cdot\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t)$$

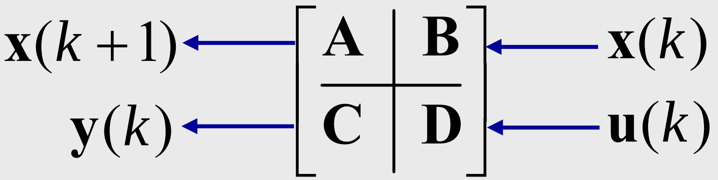


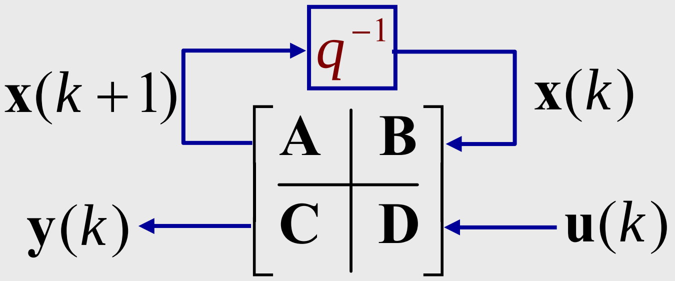


Discrete-Time

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$





$$\left[\begin{array}{c|c} \mathbf{F} & \mathbf{G} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

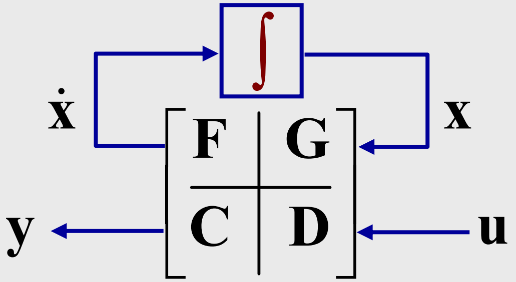
Often referred to as system (**A,B,C,D**)

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

State Space ZOH Equivalent System

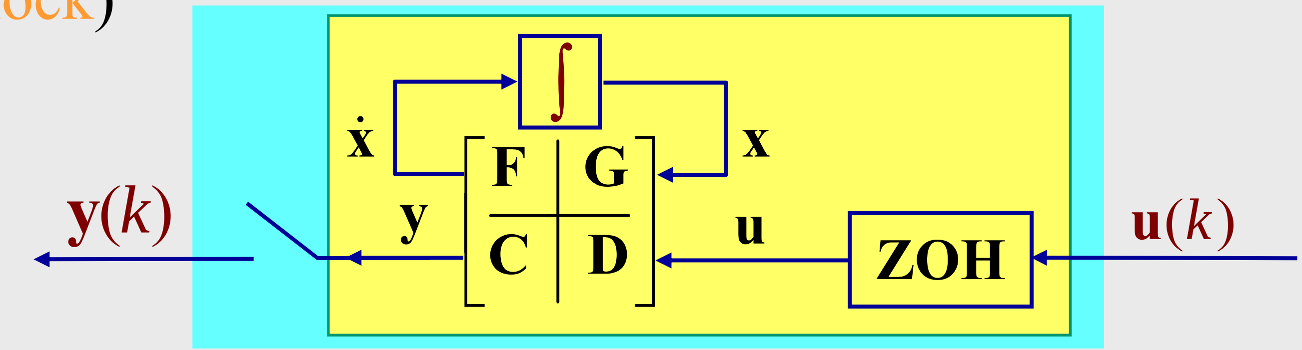
Given the C.T. system (dynamics),

$$\left[\begin{array}{c|c} \mathbf{F} & \mathbf{G} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right] \qquad \text{i.e.}$$



preceded by a ZOH with constant sampling period T

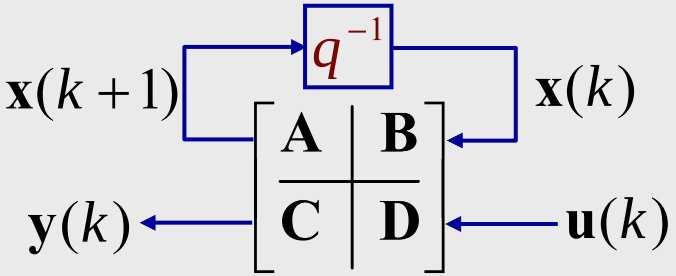
(i.e., yellow block)



Q: What is the equivalent discrete-time (D.T.) representation in green block?

$$\mathbf{x}(k+1) = \mathbf{A} \cdot \mathbf{x}(k) + \mathbf{B} \cdot \mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C} \cdot \mathbf{x}(k) + \mathbf{D} \cdot \mathbf{u}(k)$$

or



State Space ZOH Equivalent System

- **Output Equation**

  Direct sample of the output equation does not change the algebraic relationship , i.e.

$$\mathbf{y}(t) = \mathbf{C}\cdot\mathbf{x}(t) + \mathbf{D}\cdot\mathbf{u}(t) \xrightarrow[T]{\textbf{Sample}} \mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

- **State Equation**

  Obtain 1st DE from the 1st order ODE with ZOH input, i.e.

$$\begin{tikzcd}
\left\{\mathbf{u}(k),\, k \in Z\right\} \arrow[r, "\textbf{ZOH}"] & \left\{\mathbf{u}(t),\, t \in R\right\} \\
\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k) & \dot{\mathbf{x}}(t) = \mathbf{F}\cdot\mathbf{x}(t) + \mathbf{G}\cdot\mathbf{u}(t) \\
\left\{\mathbf{x}(k),\, k \in Z\right\} & \left\{\mathbf{x}(t),\, t \in R\right\} \arrow[l, "\textbf{Sample}", "T"']
\end{tikzcd}$$

***Q: What is the relationship between* (F, G) *and* (A, B)?**

Need to know the solution to the 1st order ODE!

$\Rightarrow$ Need to calculate the *state transition matrix* $\mathbf{\Phi}(t)$

C.T. State Transition Matrix

- The state transition matrix $\Phi(t)$ of a system is a $n \times n$ matrix that satisfies the homogeneous state equations $\dot{\mathbf{x}}(t) = \mathbf{F} \cdot \mathbf{x}(t)$, i.e.,

$$\frac{d}{dt}\Phi(t) = \mathbf{F} \cdot \Phi(t), \quad \Phi(0) = \mathbf{I}$$

- Let $\mathbf{x}(0)$ denote the initial state of the system at time $t = 0$, the free response (when $\mathbf{u} = 0$) of this system, for any time $t > 0$, can be calculated by

$$\mathbf{x}(t) = \Phi(t) \cdot \mathbf{x}(0)$$

- The state transition matrix $\Phi(t)$ is uniquely determined by the system matrix **F** through

$$\boxed{\Phi(t) = \mathcal{L}^{-1}\left[\left(s\mathbf{I} - \mathbf{F}\right)^{-1}\right] = e^{\mathbf{F}t}}$$

C.T. State Transition Matrix

Show that

$$\boxed{\Phi(t)=\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{F})^{-1}\right]=e^{\mathbf{F}t}}$$

Proof: (First half)

$$\dot{\mathbf{x}}(t)=\mathbf{F}\cdot\mathbf{x}(t)$$

$$\Big\downarrow \mathcal{L}[\bullet]$$

$$s\,\mathbf{X}(s)-\mathbf{x}(0)=\mathbf{F}\cdot\mathbf{X}(s)$$

$$\Rightarrow\quad \mathbf{X}(s)=(s\mathbf{I}-\mathbf{F})^{-1}\cdot\mathbf{x}(0)$$

$$\Big\downarrow \mathcal{L}^{-1}[\bullet]$$

$$\mathbf{x}(t)=\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{F})^{-1}\right]\cdot\mathbf{x}(0)$$

$$\because\quad \mathbf{x}(t)=\Phi(t)\cdot\mathbf{x}(0)\quad\Rightarrow\quad\Phi(t)=\mathcal{L}^{-1}\left[(s\mathbf{I}-\mathbf{F})^{-1}\right]$$

C.T. State Transition Matrix

Proof: (Second half)

$$\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\right]=e^{\mathbf{F}t}$$

Indirect proof:

$$e^{\mathbf{F}t}=\mathbf{I}+\mathbf{F}t+\frac{1}{2!}\mathbf{F}^2t^2+\frac{1}{3!}\mathbf{F}^3t^3+\cdots$$

$$\Rightarrow\quad \frac{d}{dt}e^{\mathbf{F}t}=\mathbf{F}+\mathbf{F}^2t+\frac{1}{2!}\mathbf{F}^3t^2+\cdots=\mathbf{F}\cdot\left(\mathbf{I}+\mathbf{F}t+\frac{1}{2!}\mathbf{F}^2t^2+\frac{1}{3!}\mathbf{F}^3t^3+\cdots\right)$$

$$\Rightarrow\quad \frac{d}{dt}e^{\mathbf{F}t}=\mathbf{F}\cdot e^{\mathbf{F}t},\quad \text{since}\quad \frac{d}{dt}\Phi(t)=\mathbf{F}\cdot\Phi(t)\quad\Rightarrow\quad \Phi(t)=e^{\mathbf{F}t}$$

Direct proof:

$$\mathcal{L}\left[e^{\mathbf{F}t}\right]=\int_0^\infty e^{\mathbf{F}t}\cdot e^{-st}\,dt=\int_0^\infty e^{-(s\mathbf{I}-\mathbf{F})t}\,dt$$

$$=(s\mathbf{I}-\mathbf{F})^{-1}e^{-(s\mathbf{I}-\mathbf{F})t}\Big|_{t=0}^{t=\infty}=(s\mathbf{I}-\mathbf{F})^{-1}$$

C.T. State Transition Matrix

- **Example**:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u=\mathbf{F}\cdot\mathbf{x}+\mathbf{B}\cdot u$$

Find the state transition matrix Φ (t).

$$\boxed{\Phi(t)=\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\right]=e^{\mathbf{F}t}}$$

$$s\mathbf{I}-\mathbf{F}=\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}=\begin{bmatrix}s & -1\\ 2 & s+3\end{bmatrix}$$

$$\left(s\mathbf{I}-\mathbf{F}\right)^{-1}=\frac{\operatorname{Adj}\left(s\mathbf{I}-\mathbf{F}\right)}{\det\left(s\mathbf{I}-\mathbf{F}\right)}=\frac{1}{(s+1)(s+2)}\begin{bmatrix}s+3 & 1\\ -2 & s\end{bmatrix}$$

$$\Phi(t)=e^{\mathbf{F}t}=\mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\right]=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

C.T. System Response

Given any initial condition $\mathbf{x}(0)$ and input $\mathbf{u}(t)$, the solution to the state equation $\dot{\mathbf{x}}(t) = \mathbf{F}\cdot\mathbf{x}(t) + \mathbf{G}\cdot\mathbf{u}(t)$ is

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{F}\cdot\mathbf{X}(s) + \mathbf{G}\cdot\mathbf{U}(s)$$

$$\Rightarrow \quad \mathbf{X}(s) = \left(s\mathbf{I}-\mathbf{F}\right)^{-1}\cdot\mathbf{x}(0) + \left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\cdot\mathbf{U}(s)$$

$$\Rightarrow \quad \mathbf{x}(t) = \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\right]\cdot\mathbf{x}(0) + \mathcal{L}^{-1}\left[\left(s\mathbf{I}-\mathbf{F}\right)^{-1}\mathbf{G}\cdot\mathbf{U}(s)\right]$$

$$= \boldsymbol{\Phi}(t)\cdot\mathbf{x}(0) + \int_0^t \boldsymbol{\Phi}(t-\tau)\mathbf{G}\cdot\mathbf{u}(\tau)\,d\tau$$

$$\Rightarrow \quad \mathbf{x}(t) = e^{\mathbf{F}t}\cdot\mathbf{x}(0) + \int_0^t e^{\mathbf{F}(t-\tau)}\mathbf{G}\cdot\mathbf{u}(\tau)\,d\tau$$

If the initial time is some non-zero value t_0,

$$\mathbf{x}(t) = e^{\mathbf{F}(t-t_0)}\cdot\mathbf{x}(t_0) + \int_{t_0}^t e^{\mathbf{F}(t-\tau)}\mathbf{G}\cdot\mathbf{u}(\tau)\,d\tau$$

Free (initial response)
Depends on **F**

Forced response
Depends both **F** and **G**

C.T. System Response

- **Example**:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u=\mathbf{F}\cdot\mathbf{x}+\mathbf{B}\cdot u$$

Find the unit step response.

Recall

$$\boldsymbol{\Phi}(t)=e^{\mathbf{F}t}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}$$

Step response

$$\mathbf{x}(t)=e^{\mathbf{F}t}\cdot\mathbf{x}(0)+\int_0^t e^{\mathbf{F}(t-\tau)}\mathbf{G}\cdot u(\tau)\,d\tau$$

$$=e^{\mathbf{F}t}\cdot\mathbf{x}(0)+\int_0^t\begin{bmatrix}2e^{-(t-\tau)}-e^{-2(t-\tau)} & e^{-(t-\tau)}-e^{-2(t-\tau)}\\ -2e^{-(t-\tau)}+2e^{-2(t-\tau)} & -e^{-(t-\tau)}+2e^{-2(t-\tau)}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}\cdot(1)\,d\tau$$

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}2e^{-t}-e^{-2t} & e^{-t}-e^{-2t}\\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t}\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}+\begin{bmatrix}\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}\\ e^{-t}-e^{-2t}\end{bmatrix}$$

State Space ZOH Equivalent System

Given $(\mathbf{F}, \mathbf{G})$ and T $\qquad$ find $(\mathbf{A}, \mathbf{B})$

$$\dot{\mathbf{x}}(t)=\mathbf{F}\cdot\mathbf{x}(t)+\mathbf{G}\cdot\mathbf{u}(t)\ \xrightarrow[\textbf{ZOH}]{\textbf{Sample } T}\ \mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{u}(k)$$

Start with what we know:

$$\mathbf{x}(t)=e^{\mathbf{F}(t-t_0)}\cdot\mathbf{x}(t_0)+\int_{t_0}^{t}e^{\mathbf{F}(t-\tau)}\mathbf{G}\cdot\mathbf{u}(\tau)\,d\tau$$

$$\boxed{\begin{array}{c}\textbf{ZOH}\\ \mathbf{u}(t)=\mathbf{u}(kT)\\ kT\le t<(k+1)T\end{array}}$$

Let $t_0=kT$ and $t=(k+1)T$, then

$$\mathbf{x}\big((k+1)T\big)=e^{\mathbf{F}T}\cdot\mathbf{x}(kT)+\int_{kT}^{(k+1)T}e^{\mathbf{F}((k+1)T-\tau)}\mathbf{G}\cdot\mathbf{u}(\tau)\,d\tau$$

$$\Rightarrow\quad \mathbf{x}\big((k+1)T\big)=e^{\mathbf{F}T}\cdot\mathbf{x}(kT)+\left[\int_{kT}^{(k+1)T}e^{\mathbf{F}((k+1)T-\tau)}\,d\tau\right]\cdot\mathbf{G}\cdot\mathbf{u}(kT)$$

Let $\eta=(k+1)T-\tau$

$$\Rightarrow\quad \mathbf{x}\big((k+1)T\big)=e^{\mathbf{F}T}\cdot\mathbf{x}(kT)+\left[\int_{T}^{0}e^{\mathbf{F}\eta}\,(-d\eta)\right]\mathbf{G}\cdot\mathbf{u}(kT)$$

$$\Rightarrow\quad \mathbf{x}(k+1)=\underbrace{e^{\mathbf{F}T}}_{\mathbf{A}(T)}\cdot\mathbf{x}(k)+\underbrace{\left[\int_{0}^{T}e^{\mathbf{F}\eta}\,d\eta\right]\mathbf{G}}_{\mathbf{B}(T)}\cdot\mathbf{u}(k)$$

State Space ZOH Equivalent System

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{F}\cdot\mathbf{x}(t)+\mathbf{G}\cdot\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}\cdot\mathbf{x}(t)+\mathbf{D}\cdot\mathbf{u}(t)\end{aligned} \quad \xrightarrow[\text{ZOH}]{\text{Sample } T} \quad \begin{aligned}\mathbf{x}(k+1) &= \mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{u}(k)\\ \mathbf{y}(k) &= \mathbf{C}\cdot\mathbf{x}(k)+\mathbf{D}\cdot\mathbf{u}(k)\end{aligned}$$

$$\boxed{\mathbf{A}(T)=\Phi(T)=e^{\mathbf{F}T}} \quad \text{and} \quad \boxed{\mathbf{B}(T)=\left(\int_0^T e^{\mathbf{F}\eta}d\eta\right)\cdot\mathbf{G}}$$

- If **F** is non-singular (no zero eigenvalue),

$$\begin{aligned}\mathbf{B}(T)=\left(\int_0^T e^{\mathbf{F}\eta}d\eta\right)\mathbf{G} &= \mathbf{F}^{-1}e^{\mathbf{F}\eta}\Big|_0^T\,\mathbf{G}=\mathbf{F}^{-1}\left(e^{\mathbf{F}T}-\mathbf{I}\right)\mathbf{G}\\ &= e^{\mathbf{F}\eta}\mathbf{F}^{-1}\Big|_0^T\,\mathbf{G}=\left(e^{\mathbf{F}T}-\mathbf{I}\right)\mathbf{F}^{-1}\mathbf{G}\end{aligned}$$

- If the sampling period is sufficiently small, then

$$\mathbf{A}\approx\mathbf{I}+\mathbf{F}\cdot T \qquad \text{and} \qquad \mathbf{B}\approx T\cdot\mathbf{G}$$

- Both **A** and **B** are functions of the sampling period T

$$\text{as} \quad T\to 0, \quad \mathbf{A}(T)\to\mathbf{I} \quad \text{and} \quad \mathbf{B}(T)\to 0$$

Does this make any sense?

State Space ZOH Equivalent System

- **Example:**

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u$$

$$\Rightarrow \quad \mathbf{A}=\Phi(T)=e^{\mathbf{F}T}=\begin{bmatrix}2e^{-T}-e^{-2T} & e^{-T}-e^{-2T}\\ -2e^{-T}+2e^{-2T} & -e^{-T}+2e^{-2T}\end{bmatrix}$$

$$\mathbf{B}=\left[\int_0^T e^{\mathbf{F}\eta}\,d\eta\right]\mathbf{G}=\int_0^T\begin{bmatrix}2e^{-\eta}-e^{-2\eta} & e^{-\eta}-e^{-2\eta}\\ -2e^{-\eta}+2e^{-2\eta} & -e^{-\eta}+2e^{-2\eta}\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}d\eta$$

$$=\begin{bmatrix}\frac{1}{2}+\frac{1}{2}e^{-2T}-e^{-T}\\ e^{-T}-e^{-2T}\end{bmatrix}$$

	$T=0.01$	$T=0.1$	$T=1$
A	$\begin{bmatrix}0.9999 & 0.0099\\ -0.0197 & 0.9703\end{bmatrix}$	$\begin{bmatrix}0.991 & 0.0861\\ -0.1722 & 0.7326\end{bmatrix}$	$\begin{bmatrix}0.6004 & 0.2325\\ -0.4651 & -0.0972\end{bmatrix}$
B	$\begin{bmatrix}0\\ 0.0099\end{bmatrix}$	$\begin{bmatrix}0.0045\\ 0.0861\end{bmatrix}$	$\begin{bmatrix}0.1998\\ 0.2325\end{bmatrix}$

State Space ZOH Equivalent System

- **Example**: (double integrator)

$$G(s)=\frac{1}{s^2} \Rightarrow \begin{aligned}\dot{x}_1&=x_2\\ \dot{x}_2&=u\end{aligned} \Rightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\cdot u$$

$$\therefore \left[\begin{array}{c|c}\mathbf{F}&\mathbf{G}\\ \hline \mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}0&1&0\\0&0&1\\ \hline 1&0&0\end{array}\right] \Rightarrow \Phi(t)=\mathcal{L}^{-1}\begin{bmatrix}1/s&1/s^2\\0&1/s\end{bmatrix}=\begin{bmatrix}1&t\\0&1\end{bmatrix}$$

$$\mathbf{A}=\Phi(T)=\begin{bmatrix}1&T\\0&1\end{bmatrix} \quad \mathbf{B}=\int_0^T e^{\mathbf{F}\eta}d\eta\cdot\mathbf{G}=\int_0^T\begin{bmatrix}1&\eta\\0&1\end{bmatrix}d\eta\cdot\mathbf{G}=\begin{bmatrix}T^2/2\\T\end{bmatrix}$$

$$\therefore \left[\begin{array}{c|c}\mathbf{A}&\mathbf{B}\\ \hline \mathbf{C}&\mathbf{D}\end{array}\right]=\left[\begin{array}{cc|c}1&T&T^2/2\\0&1&T\\ \hline 1&0&0\end{array}\right]$$

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\end{bmatrix}=\begin{bmatrix}1&T\\0&1\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}T^2/2\\T\end{bmatrix}\cdot u(k)$$

$$y(k)=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+0\cdot u(k)$$

Solution to Discrete-Time SS Equation

Given discrete-time system:

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

Initial condition:
$\mathbf{x}(0)$

Recursive solution:

At $k=0$:

$$\mathbf{x}(1) = \mathbf{A}\cdot\mathbf{x}(0) + \mathbf{B}\cdot\mathbf{u}(0)$$
$$\mathbf{x}(2) = \mathbf{A}\cdot\mathbf{x}(1) + \mathbf{B}\cdot\mathbf{u}(1)$$
$$= \mathbf{A}^2\cdot\mathbf{x}(0) + \mathbf{AB}\cdot\mathbf{u}(0) + \mathbf{B}\cdot\mathbf{u}(1)$$
$$\mathbf{x}(3) = \mathbf{A}^3\cdot\mathbf{x}(0) + \mathbf{A}^2\mathbf{B}\cdot\mathbf{u}(0) + \mathbf{AB}\cdot\mathbf{u}(1) + \mathbf{B}\cdot\mathbf{u}(2)$$
$$\vdots$$

$$\Rightarrow \quad \mathbf{x}(k) = \mathbf{A}^k\cdot\mathbf{x}(0) + \sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{B}\cdot\mathbf{u}(j)$$
$$\mathbf{y}(k) = \mathbf{CA}^k\cdot\mathbf{x}(0) + \mathbf{C}\cdot\sum_{j=0}^{k-1}\mathbf{A}^{k-1-j}\mathbf{B}\cdot\mathbf{u}(j) + \mathbf{D}\cdot\mathbf{u}(k)$$

Solution to Discrete-Time SS Equation

Given discrete-time system:

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k)=\mathbf{C}\cdot\mathbf{x}(k)+\mathbf{D}\cdot\mathbf{u}(k)$$

Initial condition:

$$\mathbf{x}(0)$$

$\mathcal{Z}[\bullet]$

***z*-Transform solution:**

$$z\cdot\mathbf{X}(z)-z\cdot\mathbf{x}(0)=\mathbf{A}\cdot\mathbf{X}(z)+\mathbf{B}\cdot\mathbf{U}(z)$$

$$\left(z\mathbf{I}-\mathbf{A}\right)\cdot\mathbf{X}(z)=z\cdot\mathbf{x}(0)+\mathbf{B}\cdot\mathbf{U}(z)$$

$$\mathbf{X}(z)=\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\cdot z\cdot\mathbf{x}(0)+\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}\cdot\mathbf{U}(z)$$

$$\mathbf{Y}(z)=\underbrace{\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}z\cdot\mathbf{x}(0)}_{\text{Free Response}}+\underbrace{\Big[\underbrace{\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}}_{\text{Transfer Function}}\Big]\cdot\mathbf{U}(z)}_{\text{Forced Response}}$$

$\mathcal{Z}^{-1}[\bullet]$

$$\mathbf{x}(k)=\mathcal{Z}^{-1}\left[\mathbf{X}(z)\right]=\mathcal{Z}^{-1}\left[\left(z\mathbf{I}-\mathbf{A}\right)^{-1}z\right]\cdot\mathbf{x}(0)+\mathcal{Z}^{-1}\left[\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}\cdot\mathbf{U}(z)\right]$$

$$\Rightarrow\qquad \mathbf{A}^{k}=\mathcal{Z}^{-1}\left[\left(z\mathbf{I}-\mathbf{A}\right)^{-1}z\right]$$ **State transition matrix**

D.T. State Transition Matrix

- The state transition matrix $\psi(k)$ of a system is an $n \times n$ matrix that satisfies its own homogeneous state equations, i.e.,

$$\boldsymbol{\Psi}(k+1) = \mathbf{A} \cdot \boldsymbol{\Psi}(k), \quad \boldsymbol{\Psi}(0) = \mathbf{I}$$

- Let $\mathbf{x}(0)$ denote the initial state of the system at time $k = 0$, the free response (when $\mathbf{u} = 0$) of this system, for any time $k > 0$, can be calculated by

$$\mathbf{x}(k) = \boldsymbol{\Psi}(k) \cdot \mathbf{x}(0)$$

- The state transition matrix $\psi(k)$ is uniquely determined by the system matrix $\mathbf{A}$ through

$$\boxed{\boldsymbol{\Psi}(k) = \mathcal{Z}^{-1}\left[\left(z\mathbf{I} - \mathbf{F}\right)^{-1} z\right] = \mathbf{A}^k}$$

From State Space to Transfer Function

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

Take *z*-transform and assume *zero I.C.*

$$\mathbf{Y}(z) = \left[\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}\right]\cdot\mathbf{U}(z)$$

$$\mathbf{G}(z) = \left[\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}\right] = \frac{\mathbf{C}\cdot\mathrm{Adj}\left(z\mathbf{I}-\mathbf{A}\right)\cdot\mathbf{B}}{\underbrace{\det\left(z\mathbf{I}-\mathbf{A}\right)}_{\text{Characteristic polynomial}}} + \mathbf{D}$$

Frequency response:

$$\mathbf{G}(e^{j\omega T}) = \mathbf{C}\left(e^{j\omega T}\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}$$

Poles of the system are the roots of the characteristic equation:

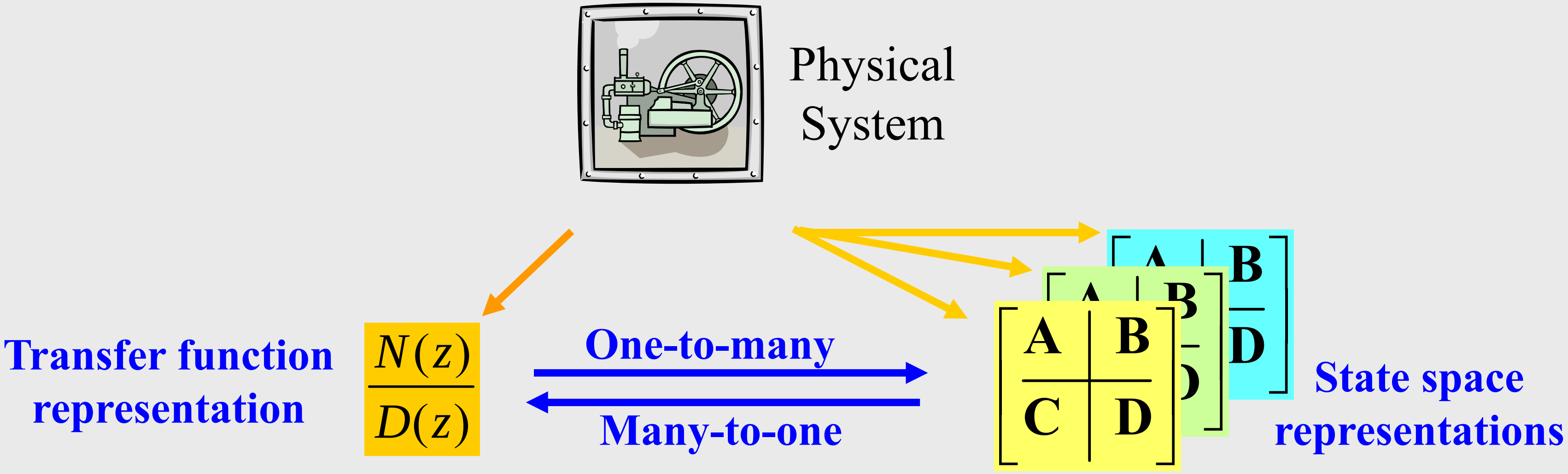
$$\det\left(z\mathbf{I}-\mathbf{A}\right) = 0$$

i.e., eigenvalues of $\mathbf{A}$

$$\det\left(\lambda\mathbf{I}-\mathbf{A}\right) = 0$$

State Space to Transfer Function

- The eigenvalues of the system matrix **A** are the poles of the system.



Physical System

Transfer function representation

$$\frac{N(z)}{D(z)}$$

One-to-many

Many-to-one

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array}\right]$$

State space representations

- Given a state space representation, assuming no pole/zero cancellation, the corresponding transfer function representation is unique.
- Given a minimal (no pole/zero cancellation) transfer function representation, the corresponding state space representation is not unique. All the state space representations of minimum dimension are related by similarity transformations.

Similarity Transformation

- Given one state space representation, any nonsingular linear transformation of that state, such as $\mathbf{x}_{NEW} = \mathbf{T}\mathbf{x}$ ($\mathbf{T}$ is nonsingular), is called a similarity transform of the original representation.

Given:

$$\mathbf{x}(k+1) = \mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\cdot\mathbf{x}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

Let

$$\mathbf{x}_{NEW} = \mathbf{T}\cdot\mathbf{x}$$

$$\mathbf{x}_{NEW}(k+1) = \mathbf{T}\cdot\mathbf{x}(k+1) = \mathbf{T}\cdot\left(\mathbf{A}\cdot\mathbf{x}(k) + \mathbf{B}\cdot\mathbf{u}(k)\right)$$

$$= \mathbf{TAT}^{-1}\cdot\mathbf{x}_{NEW}(k) + \mathbf{T}\cdot\mathbf{B}\cdot\mathbf{u}(k)$$

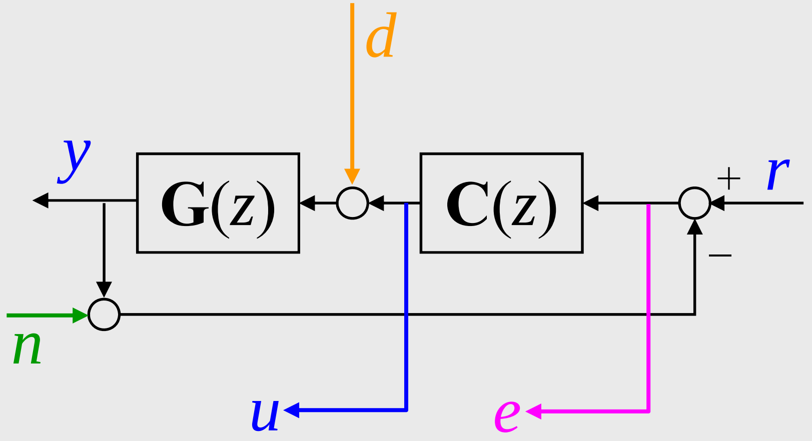
Then

$$\mathbf{x}_{NEW}(k+1) = \mathbf{TAT}^{-1}\cdot\mathbf{x}_{NEW}(k) + \mathbf{TB}\cdot\mathbf{u}(k)$$
$$\mathbf{y}(k) = \mathbf{CT}^{-1}\cdot\mathbf{x}_{NEW}(k) + \mathbf{D}\cdot\mathbf{u}(k)$$

is *another state space representation* of the same physical system

State Space Representation

- Uniform framework to analyze system dynamics (structure) for different purposes – efficient representation for multiple-input-multiple-output (MIMO) systems



$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\begin{bmatrix}\mathbf{B} & \mathbf{B}_d & \mathbf{B}_n\end{bmatrix}\begin{bmatrix} r(k)\\ d(k)\\ n(k)\end{bmatrix}$$

$$\begin{bmatrix} y(k)\\ u(k)\\ e(k)\end{bmatrix}=\begin{bmatrix}\mathbf{C}\\ \mathbf{C}_u\\ -\mathbf{C}\end{bmatrix}\cdot\mathbf{x}(k)+\begin{bmatrix}\mathbf{D} & \mathbf{D}_d & \mathbf{D}_n\\ \mathbf{D}_{ur} & \mathbf{D}_{ud} & \mathbf{D}_{un}\\ 1 & 0 & -1\end{bmatrix}\cdot\begin{bmatrix} r(k)\\ d(k)\\ n(k)\end{bmatrix}$$

While any signal in the system can be extracted to become an output, the system’s state matrix stays the same.

- Numerical robustness – can select transformation that results in numerically stable representation for computation.

Non-uniqueness of State Space Representation

Two state space representations

$$\mathbf{x}(k+1)=\mathbf{A}\cdot\mathbf{x}(k)+\mathbf{B}\cdot\mathbf{u}(k) \qquad \mathbf{x}_2(k+1)=\mathbf{A}_2\cdot\mathbf{x}_2(k)+\mathbf{B}_2\cdot\mathbf{u}(k)$$

$$\mathbf{y}(k)=\mathbf{C}\cdot\mathbf{x}(k)+\mathbf{D}\cdot\mathbf{u}(k) \qquad \mathbf{y}(k)=\mathbf{C}_2\cdot\mathbf{x}_2(k)+\mathbf{D}_2\cdot\mathbf{u}(k)$$

where $\mathbf{x}_2=\mathbf{T}\mathbf{x}$, and

$$\mathbf{A}_2=\mathbf{T}\mathbf{A}\mathbf{T}^{-1},\quad \mathbf{B}_2=\mathbf{T}\mathbf{B},\quad \mathbf{C}_2=\mathbf{C}\mathbf{T}^{-1},\quad \mathbf{D}_2=\mathbf{D}$$

Let

$$\mathbf{G}(z)=\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D} \qquad \mathbf{G}_2(z)=\mathbf{C}_2\left(z\mathbf{I}-\mathbf{A}_2\right)^{-1}\mathbf{B}_2+\mathbf{D}_2$$

Then

$$\begin{aligned}\mathbf{G}_2(z)=\mathbf{C}_2\left(z\mathbf{I}-\mathbf{A}_2\right)^{-1}\mathbf{B}_2+\mathbf{D}_2&=\mathbf{C}\mathbf{T}^{-1}\left(z\mathbf{I}-\mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\mathbf{T}\mathbf{B}+\mathbf{D}\\ &=\mathbf{C}\left[\mathbf{T}^{-1}\left(z\mathbf{I}-\mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)^{-1}\left(\mathbf{T}^{-1}\right)^{-1}\right]\mathbf{B}+\mathbf{D}\\ &=\mathbf{C}\left[\mathbf{T}^{-1}\left(z\mathbf{I}-\mathbf{T}\mathbf{A}\mathbf{T}^{-1}\right)\mathbf{T}\right]^{-1}\mathbf{B}+\mathbf{D}=\mathbf{C}\left(z\mathbf{I}-\mathbf{A}\right)^{-1}\mathbf{B}+\mathbf{D}=\mathbf{G}(z)\end{aligned}$$

For a given input/output relationship, there can be many different state space representations (similar to each other)

From Transfer Function to State Space

- Since the SS representation of a system is not unique, given the TF of a discrete-time system, there can be infinite number of state space representations. In the following discussion, we will introduce two canonical representations that are most often used.

- The pulse transfer function of the system is assumed known and is written as

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0+b_1z^{-1}+b_2z^{-2}+\cdots+b_nz^{-n}}{1+a_1z^{-1}+a_2z^{-2}+\cdots+a_nz^{-n}}$$

$$=\frac{b_0z^n+b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n}{z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n}$$

$$=b_0+\frac{(b_1-a_1b_0)z^{n-1}+(b_2-a_2b_0)z^{n-2}+\cdots+(b_n-a_nb_0)}{z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n}$$

Controllable Canonical Forms

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix}}_{\mathbf{x}_C(k+1)} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}}_{\mathbf{A}_C} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}}_{\mathbf{x}_C(k)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{B}_C} \cdot u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} (b_n - a_n b_0) & (b_{n-1} - a_{n-1} b_0) & \cdots & (b_1 - a_1 b_0) \end{bmatrix}}_{\mathbf{C}_C} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}}_{\mathbf{x}_C(k)} + \underbrace{b_0}_{\mathbf{D}_C} \cdot u(k)$$

If $b_0 = 0$, then

$$y(k) = \underbrace{\begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix}}_{\mathbf{C}_C} \cdot \mathbf{x}_C(k)$$

System is strictly proper!

Controllable Canonical Form

Let

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0 z^n+b_1 z^{n-1}+b_2 z^{n-2}+\cdots+b_n}{z^n+a_1 z^{n-1}+a_2 z^{n-2}+\cdots+a_n}\cdot\frac{X_1(z)}{X_1(z)}$$

Choose

$$\begin{aligned}
z\,X_1(z)&=X_2(z) &&\rightarrow && x_1(k+1)=x_2(k)\\
z\,X_2(z)&=X_3(z) &&\rightarrow && x_2(k+1)=x_3(k)\\
z\,X_3(z)&=X_4(z) &&\rightarrow && x_3(k+1)=x_4(k)\\
& &&\vdots && \\
z\,X_{n-1}(z)&=X_n(z) &&\rightarrow && x_{n-1}(k+1)=x_n(k)
\end{aligned}$$

From the denominator

$$U(z)=\left(z^n+a_1 z^{n-1}+a_2 z^{n-2}+\cdots+a_n\right)\cdot X_1(z)$$

$$\Rightarrow \quad z^n X_1=\left(-a_1 z^{n-1}-a_2 z^{n-2}-\cdots-a_n\right)\cdot X_1(z)+U(z)$$

$$\Rightarrow \quad x_n(k+1)=-a_1 x_n(k)-a_2 x_{n-1}(k)-\cdots-a_n x_1(k)+u(k)$$

Controllable Canonical Form

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0z^n+b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n}{z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n}\cdot\frac{X_1(z)}{X_1(z)}$$

From the numerator

$$Y(z)=\left(b_0\,z^n+b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n\right)\cdot X_1(z)$$

$$Y(z)=b_0\overbrace{\left[\left(-a_1z^{n-1}-a_2z^{n-2}-\cdots-a_n\right)X_1+U(z)\right]}$$

$$+\left(b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n\right)X_1$$

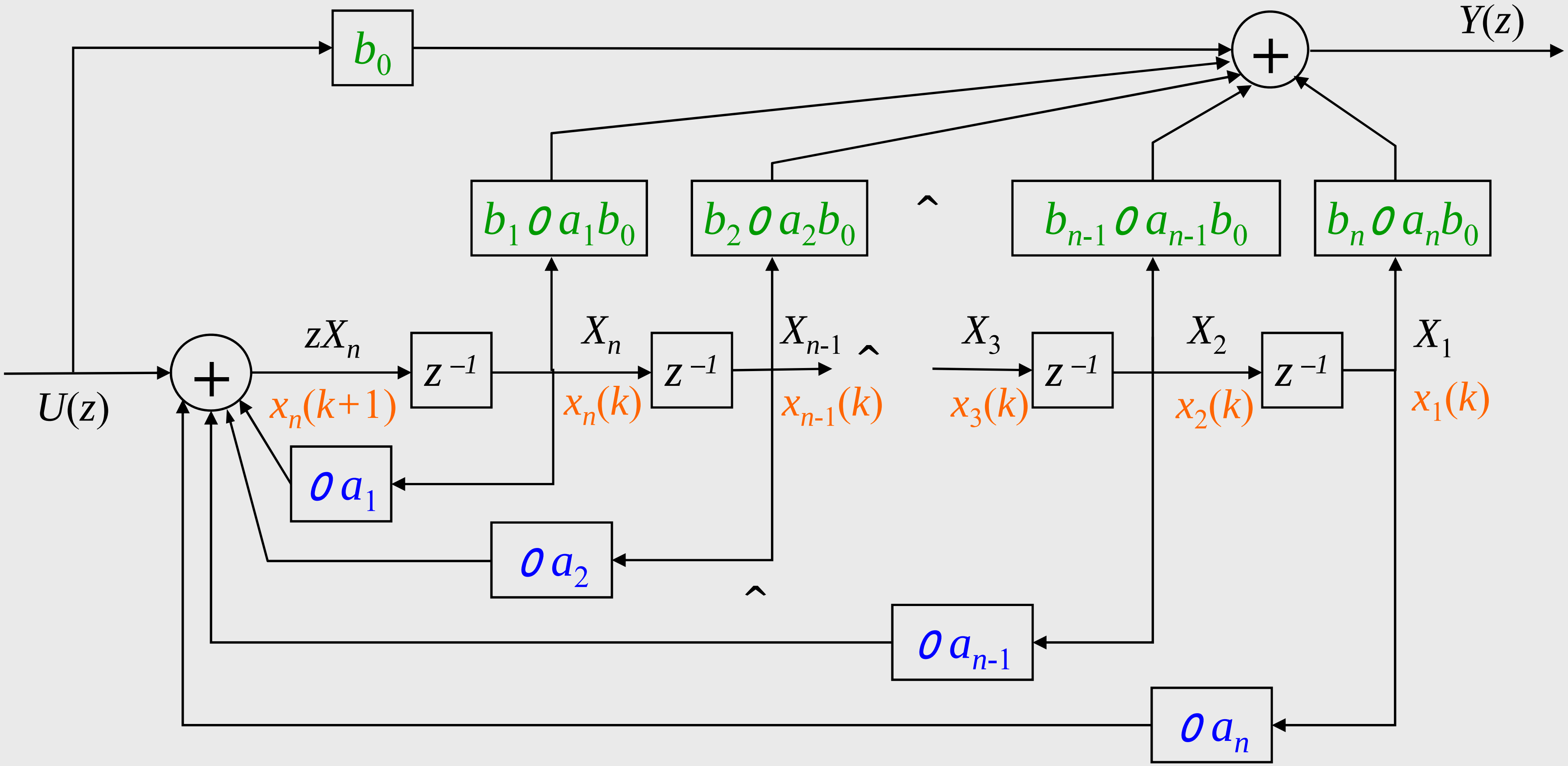
$$\Rightarrow\quad Y(z)=\left(b_n-a_nb_0\right)X_1(z)+\left(b_{n-1}-a_{n-1}b_0\right)X_2(z)$$

$$+\cdots+\left(b_1-a_1b_0\right)X_n(z)+b_0U(z)$$

$$y(k)=\left(b_n-a_nb_0\right)x_1(k)+\left(b_{n-1}-a_{n-1}b_0\right)x_2(k)+\cdots+\left(b_1-a_1b_0\right)x_n(k)$$

$$+b_0u(k)$$

Controllable Canonical Form



Controllable Canonical Form

- **Example:**

$$G(z)=\frac{z^2+2z+1}{z^2+5z+6}$$

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0z^n+b_1z^{n-1}+b_2z^{n-2}+\cdots+b_n}{z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n}$$

$$\Rightarrow \quad b_0=1 \quad b_1=2 \quad b_2=1$$

$$a_1=5 \quad a_2=6$$

$$\Downarrow$$

$$\begin{bmatrix}x_1(k+1)\\x_2(k+1)\end{bmatrix}=\begin{bmatrix}0&1\\-6&-5\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}\cdot u(k)$$

$$y(k)=\begin{bmatrix}-5&-3\end{bmatrix}\begin{bmatrix}x_1(k)\\x_2(k)\end{bmatrix}+u(k)$$

$$\underbrace{\begin{bmatrix}x_1(k+1)\\x_2(k+1)\\\vdots\\x_{n-1}(k+1)\\x_n(k+1)\end{bmatrix}}_{\mathbf{x}_C(k+1)}=\underbrace{\begin{bmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&0&1\\-a_n&-a_{n-1}&-a_{n-2}&\cdots&-a_2&-a_1\end{bmatrix}}_{\mathbf{A}_C}\underbrace{\begin{bmatrix}x_1(k)\\x_2(k)\\\vdots\\x_{n-1}(k)\\x_n(k)\end{bmatrix}}_{\mathbf{x}_C(k)}+\underbrace{\begin{bmatrix}0\\0\\\vdots\\0\\1\end{bmatrix}}_{\mathbf{B}_C}\cdot u(k)$$

$$y(k)=\underbrace{\begin{bmatrix}(b_n-a_nb_0)&(b_{n-1}-a_{n-1}b_0)&\cdots&(b_1-a_1b_0)\end{bmatrix}}_{\mathbf{C}_C}\underbrace{\begin{bmatrix}x_1(k)\\x_2(k)\\\vdots\\x_{n-1}(k)\\x_n(k)\end{bmatrix}}_{\mathbf{x}_C(k)}+\underbrace{b_0}_{\mathbf{D}_C}\cdot u(k)$$

Observable Canonical Form

$$\underbrace{\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ \vdots \\ x_{n-1}(k+1) \\ x_n(k+1) \end{bmatrix}}_{\mathbf{x}_\mathrm{O}(k+1)} = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -a_n \\ 1 & 0 & \cdots & 0 & 0 & -a_{n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & -a_2 \\ 0 & 0 & \cdots & 0 & 1 & -a_1 \end{bmatrix}}_{\mathbf{A}_\mathrm{O}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}}_{\mathbf{x}_\mathrm{O}(k)} + \underbrace{\begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix}}_{\mathbf{B}_\mathrm{O}} \cdot u(k)$$

$$y(k) = \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{\mathbf{C}_\mathrm{O}} \underbrace{\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n-1}(k) \\ x_n(k) \end{bmatrix}}_{\mathbf{x}_\mathrm{O}(k)} + \underbrace{b_0}_{\mathbf{D}_\mathrm{O}} \cdot u(k)$$

Transposed SISO System

Let

$$G(z) = \frac{Y(z)}{U(z)} = \mathbf{C}_C \left(z\mathbf{I} - \mathbf{A}_C\right)^{-1} \mathbf{B}_C + \mathbf{D}_C \quad \text{SISO system}$$

Then

$$G^T(z) = G(z) = \left[\mathbf{C}_C \left(z\mathbf{I} - \mathbf{A}_C\right)^{-1} \mathbf{B}_C + \mathbf{D}_C\right]^T$$

$$= \mathbf{B}_C{}^T \left(z\mathbf{I} - \mathbf{A}_C{}^T\right)^{-1} \mathbf{C}_C{}^T + \mathbf{D}_C{}^T$$

$$= \mathbf{C}_O \left(z\mathbf{I} - \mathbf{A}_O\right)^{-1} \mathbf{B}_O + \mathbf{D}_O$$

- For an SISO system, if (**A**, **B**, **C**, **D**) is a valid representation of a system, then ($\mathbf{A}^T$, $\mathbf{C}^T$, $\mathbf{B}^T$, $\mathbf{D}^T$) is also a valid representation of the same system
- For an SISO system, if the representation (**A**, **B**, **C**, **D**) is in the controllable canonical form, then ($\mathbf{A}^T$, $\mathbf{C}^T$, $\mathbf{B}^T$, $\mathbf{D}^T$) is in the observable canonical form