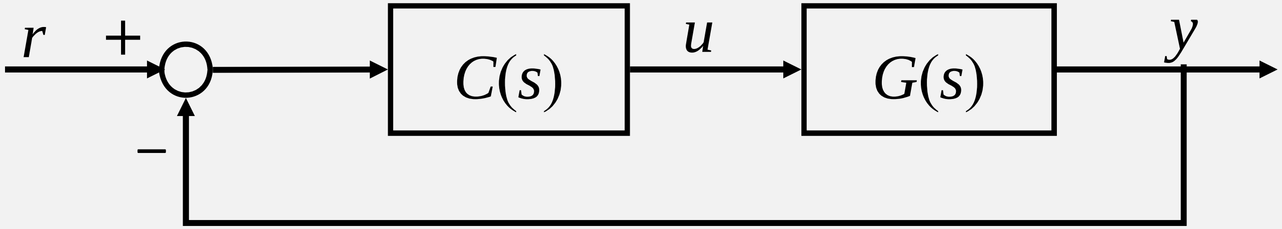
The z-Transform and Difference Equations

- The z-Transform
  - Definition
  - Properties
- Inverse z-Transform
- Solving Linear Difference Equations Using z-Transform
- Pulse Transfer Function
- Impulse Response Sequence
- Frequency Response of Discrete-Time Systems

Z-Transform

- The counter-part of Laplace transform (*used in the continuous-time domain*) in the *discrete-time domain*.
- Why Laplace transform?
  - Differentiation/integration $\rightarrow$ algebraic operations
  - Convolution relationships between signals are transformed into multiplications

Why Analyze Signals/Systems in the Transformed Domain?



In time domain:

$$y(t)=\int_0^t g(t-\tau_1)u(\tau_1)\,d\tau_1=\int_0^t g(\tau_1)u(t-\tau_1)\,d\tau_1$$

$$u(t)=\int_0^t c(t-\tau_2)\big[r(\tau_2)-y(\tau_2)\big]\,d\tau_2=\int_0^t c(\tau_2)\big[r(t-\tau_2)-y(t-\tau_2)\big]\,d\tau_2$$

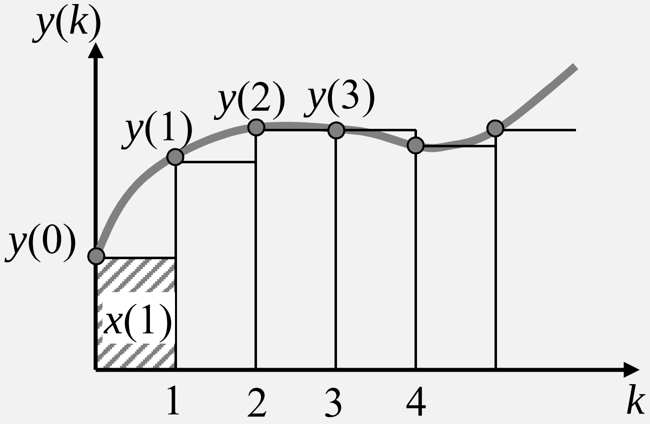
$$\Rightarrow \quad y(t)=\int_0^t g(t-\tau_1)\left[\int_0^{\tau_1} c(\tau_1-\tau_2)\big[r(\tau_2)-y(\tau_2)\big]\,d\tau_2\right]d\tau_1$$

$$=\int_0^t\left[\int_0^{t-\tau_1}[r(t-\tau_1-\tau_2)-y(t-\tau_1-\tau_2)]\cdot c(\tau_2)\,d\tau_2\right]\cdot p(\tau_1)\,d\tau_1$$

In transformed domain:

$$Y(s)=\frac{G(s)C(s)}{1+G(s)C(s)}\cdot R(s)$$

Solution to Difference Equations



$\Rightarrow \dot{x}(k-1) = y(k-1)$

To solve $\dot{x} = y$ numerically using Euler approximation: $\frac{x(k) - x(k-1)}{T} = y(k-1)$

$$\Rightarrow \quad x(k) = x(k-1) + y(k-1) \cdot T$$

difference equation

To find a closed form solution of the difference equation:

$$x(1) = x(0) + y(0) \cdot T$$
$$x(2) = x(1) + y(1) \cdot T$$
$$x(3) = x(2) + y(2) \cdot T$$
$$\vdots$$
$$+) \quad x(k) = x(k-1) + y(k-1) \cdot T$$

$$x(k) = x(0) + \left(\sum_{j=0}^{k-1} y(j) \right) \cdot T$$

Solution to the difference eq.

For more complicated difference equation, a systematic approach is needed!!!

Definition of z-Transform

- The **two-sided *z*-transform** of a sampled sequence $x(kT)$ or $x(k)$, where k is an integer and T is the sampling period, is defined by

$$X(z) = \mathcal{Z}\left[x^*(k)\right] = \mathcal{Z}\left[x(kT)\right] = \mathcal{Z}\left[x(k)\right] = \sum_{k=-\infty}^{\infty} x(kT) z^{-k} = \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

- In the **one-sided *z*-transform**, it is assumed that $x(kT) = x(k) = 0$ for $k < 0$. Then,

$$X(z) = \mathcal{Z}\left[x(k)\right] = \sum_{k=0}^{\infty} x(k) z^{-k}$$

Note: the complex variable z must be selected such that the infinite series converges

Recall: the z transform can be obtained by the $\mathcal{L}$-transformation of the periodically sampled signal in time-domain using pulse train by letting

$$X^*(s) = \sum_{k=-\infty}^{\infty} x(kT) \cdot e^{-kTs} \xrightarrow{z=e^{Ts}} X(z) = \sum_{k=-\infty}^{\infty} x(kT) \cdot z^{-k}$$

Examples of Computing z Transform

- Unit **Step** function

$$u(k)=\begin{cases}1 & k\ge 0\\ 0 & k<0\end{cases}$$

$$U(z)=\mathcal{Z}\left[u(k)\right]=\sum_{k=0}^{\infty}1\cdot z^{-k}=1+z^{-1}+z^{-2}+\cdots$$

$$z^{-1}\cdot U(z) = \sum_{k=1}^{\infty}1\cdot z^{-k} = z^{-1}+z^{-2}+z^{-3}\cdots$$

$$(1-z^{-1})\cdot U(z) = 1$$

$$U(z)=\frac{1}{1-z^{-1}}=\frac{z}{z-1} \qquad \text{for } \left|z^{-1}\right|<1 \ \left(\text{or } |z|>1\right)$$

Radius of convergence

Example of Computing z Transform

- Unit **Ramp** function

$$u(k)=\begin{cases} kT & k\geq 0 \\ 0 & k<0 \end{cases}$$

$$U(z)=\mathcal{Z}\left[u(k)\right]=\sum_{k=0}^{\infty} kT\cdot z^{-k}=T(z^{-1}+2z^{-2}+3z^{-3}+\cdots)$$

$$U(z)=T(z^{-1}+2z^{-2}+3z^{-3}+\cdots)$$

$$z^{-1}\cdot U(z)=T(\qquad z^{-2}+2z^{-3}+3z^{-4}+\cdots)$$

$$(1-z^{-1})\cdot U(z)=T\cdot(z^{-1}+z^{-2}+z^{-3}+\cdots)$$

$$=T\cdot z^{-1}\cdot(1+z^{-1}+z^{-2}+\cdots)$$

$$U(z)=T\frac{z^{-1}}{(1-z^{-1})^{2}}=\frac{Tz}{(z-1)^{2}}$$

Examples of Computing z Transform

- **Polynomials**

$$x(k) = \begin{cases} a^k & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$X(z) = \mathcal{Z}\left[x(k)\right] = \sum_{k=0}^{\infty} \left(a^k\right) \cdot z^{-k} = \sum_{k=0}^{\infty} \left(z/a\right)^{-k} = \frac{1}{1-\left(z/a\right)^{-1}} = \frac{z}{z-a}$$

Radius of convergence: $\left|z/a\right| > 1 \quad \Rightarrow \quad |z| > |a|$

- **Scaling**: if $X(z)$ is the z transform of $x(k)$, then the z transform of $a^k \cdot x(k)$ is given by $X(z/a)$, i.e.,

$$\mathcal{Z}\left[a^k \cdot x(k)\right] = \sum_{k=0}^{\infty} a^k x(k) \cdot z^{-k} = \sum_{k=0}^{\infty} x(k) \left(\frac{z}{a}\right)^{-k} = X\left(z/a\right)$$

Examples of Computing z Transform

- **Exponential** functions

$$x(k) = \begin{cases} e^{-akT} & k \geq 0 \\ 0 & k < 0 \end{cases}$$

$$X(z) = \mathcal{Z}[x(k)] = \sum_{k=0}^{\infty} e^{-akT} \cdot z^{-k} = \sum_{k=0}^{\infty} \left(e^{aT} z\right)^{-k}$$

$$\text{Recall} \quad \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1 - z^{-1}} \quad \text{if } |z| > 1$$

$$\Rightarrow \sum_{k=0}^{\infty} \left(e^{aT} z\right)^{-k} = \frac{1}{1 - \left(e^{aT} z\right)^{-1}} = \frac{z}{z - e^{-aT}} \quad \text{for} \quad |z| > e^{-aT}$$

$$X(z) = \frac{z}{z - e^{-aT}} \quad \text{for} \quad |z| > e^{-aT}$$

Examples of Computing z Transform

- **Sinusoidal** functions

$$x(k)=\begin{cases}\sin(\omega kT) & k\ge 0\\ 0 & k<0\end{cases}$$

$$X(z)=\mathcal{Z}\left[x(k)\right]=\sum_{k=0}^{\infty}\sin(\omega kT)\cdot z^{-k}=\sum_{k=0}^{\infty}\frac{e^{j\omega kT}-e^{-j\omega kT}}{2j}\cdot z^{-k}$$

$$=\frac{1}{2j}\left(\frac{1}{1-e^{j\omega T}z^{-1}}-\frac{1}{1-e^{-j\omega T}z^{-1}}\right)$$

$$=\frac{1}{2j}\frac{(e^{j\omega T}-e^{-j\omega T})z^{-1}}{1-(e^{j\omega T}+e^{-j\omega T})z^{-1}+z^{-2}}$$

$$=\frac{z^{-1}\sin(\omega T)}{1-2z^{-1}\cos(\omega T)+z^{-2}}=\frac{z\cdot\sin(\omega T)}{z^{2}-2z\cdot\cos(\omega T)+1}\quad \text{for } |z|>1$$

z Transform Table

$f(t), t \geq 0$	$F(s)$	$f(kT), k \geq 0$	$F(z)$	
–	–	$\begin{cases} 1, k=0 \\ 0, k \neq 0 \end{cases}$	1	
–	–	$\begin{cases} 1, k=n \\ 0, k \neq n \end{cases}$	z^{-n}	
1	$\frac{1}{s}$	1	$\frac{z}{z-1}$	
t	$\frac{1}{s^2}$	kT	$\frac{Tz}{(z-1)^2}$	
$\frac{1}{2}t^2$	$\frac{1}{s^3}$	$\frac{1}{2}(kT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$	
e^{-at}	$\frac{1}{s+a}$	e^{-akT}	$\frac{z}{z-e^{-aT}}$	
te^{-at}	$\frac{1}{(s+a)^2}$	$(kT)e^{-akT}$	$\frac{Te^{-aT}z}{(z-e^{-aT})^2}$	$\cdots$

Properties of the *z*-Transform

- **Linearity** – z Transformation is a linear transformation

Let $X(z)=\mathcal{Z}[x(k)]$ and $Y(z)=\mathcal{Z}[y(k)]$, then for any $a,b\in\mathcal{R}$

$$\begin{cases}\mathcal{Z}[a\cdot x(k)]=a\cdot\mathcal{Z}[x(k)]=a\cdot X(z)\\ \mathcal{Z}[a\cdot x(k)+b\cdot y(k)]=a\cdot\mathcal{Z}[x(k)]+b\cdot\mathcal{Z}[y(k)]=a\cdot X(z)+b\cdot Y(z)\end{cases}$$

Proof:

$$\mathcal{Z}[a\cdot x(k)]=\sum_{k=0}^{\infty}a\cdot x(k)\cdot z^{-k}=a\cdot\left(\sum_{k=0}^{\infty}x(k)\cdot z^{-k}\right)$$

$$=a\cdot\mathcal{Z}[x(k)]=a\cdot X(z)$$

The other equation can be proved using the same approach.

Properties of the *z*-Transform

- **Time Shift** with one-sided z-transform

If $x(k) = 0$ for $k < 0$ and $x(k)$ has the z-transform $X(z)$, then

$$\mathcal{Z}[x(k-d)] = z^{-d} \cdot X(z)$$

$$\mathcal{Z}[x(k+d)] = z^{d} \cdot \left[X(z) - \sum_{j=0}^{d-1} x(j) \cdot z^{-j} \right] = z^{d} \cdot X(z) - \sum_{i=1}^{d} x(d-i) \cdot z^{i}$$

$$= z^{d} X(z) - z^{d} x(0) - z^{d-1} x(1) - z^{d-2} x(2) - \cdots - z \cdot x(d-1)$$

Proof:

$$\mathcal{Z}[x(k-d)] = \sum_{k=0}^{\infty} x(k-d) \cdot z^{-k}$$

let $\boldsymbol{k - d = j}$

$$= \sum_{j=-d}^{\infty} x(j) \cdot z^{-(j+d)} = z^{-d} \sum_{j=-d}^{\infty} x(j) \cdot z^{-j}$$

$$= z^{-d} \sum_{j=0}^{\infty} x(j) \cdot z^{-j} = z^{-d} \cdot X(z)$$

Why?

Properties of the *z*-Transform

Time Shift

- Example:

$$\mathcal{Z}\left[x(k+1)\right] = z \cdot X(z) - z \cdot x(0)$$

$$\mathcal{Z}\left[x(k-1)\right] = z^{-1} \cdot X(z)$$

- Since $z^{-d}\,X(z)$ is the z transform for $x(k-d)$ and that $z^{d}\,X(z)$ is the z transform for $x(k+d)$ for zero initial conditions, it seems like that when a z transform is multiplied by z (or z^{-1}) it is equivalent to shifting the entire time sequence forward (or backward) by one sample instance. Hence,

Define:

One step delay operator q^{-1} $\Rightarrow$ $q^{-1} \cdot x(k) = x(k-1)$

One step advance operator q $\Rightarrow$ $q \cdot x(k) = x(k+1)$

Note: Both q^{-1} and q operates on the entire time sequence $x(k)$ and not just the value at some specific sampling instance.

Properties of the z-Transform

- **Initial Value Theorem (IVT)**

  If the z-transform of $x(k)$ is $X(z)$ and if $\lim_{z\to\infty} X(z)$ exists, then the initial value of $x(k)$ (i.e., $x(0)$) is

$$x(0) = \lim_{z\to\infty} X(z)$$

Proof:

$$X(z) = \sum_{k=0}^{\infty} x(k)\cdot z^{-k} = x(0) + x(1)\cdot z^{-1} + x(2)\cdot z^{-2} + \cdots$$

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\left[x(0) + x(1)\cdot \underbrace{z^{-1}}_{\to 0} + x(2)\cdot \underbrace{z^{-2}}_{\to 0} + \cdots\right]$$

$$\Rightarrow \quad \lim_{z\to\infty} X(z) = x(0)$$

Properties of the z-Transform

- **Final Value Theorem (FVT)**

If the z-transform of $x(k)$ is $X(z)$ and if $\lim_{k\to\infty} x(k)$ exists, then

$$x(\infty) = \lim_{k\to\infty} x(k) = \lim_{z\to 1}\left[(z-1)X(z)\right]$$

Proof:

$$\mathcal{Z}\left[x(k+1)-x(k)\right] = \sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right]z^{-k} = \mathcal{Z}\left[x(k+1)\right]-\mathcal{Z}\left[x(k)\right]$$

$$= z\cdot X(z) - z\cdot x(0) - X(z) = (z-1)X(z) - z\cdot x(0)$$

$$\lim_{z\to 1}\left(\sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right]z^{-k}\right) = \lim_{z\to 1}\left[(z-1)X(z) - z\cdot x(0)\right]$$

$$= \sum_{k=0}^{\infty}\left[x(k+1)-x(k)\right] \qquad = \lim_{z\to 1}\left[(z-1)X(z)\right] - x(0)$$

$$= \lim_{k\to\infty} x(k) - x(0)$$

(the $x(0)$ terms on both sides are struck out)

Properties of the *z*-Transform

- **Convolution**
  - Discrete convolution in the time domain is equivalent to multiplication in the z domain.

Define: Convolution operator $\otimes$

$$g \otimes u(k) = \sum_{j=0}^{k} g(k-j) \cdot u(j) = \sum_{j=0}^{k} g(j) \cdot u(k-j)$$

then

$$\mathcal{Z}\left[g \otimes u(k)\right] = G(z) \cdot U(z)$$

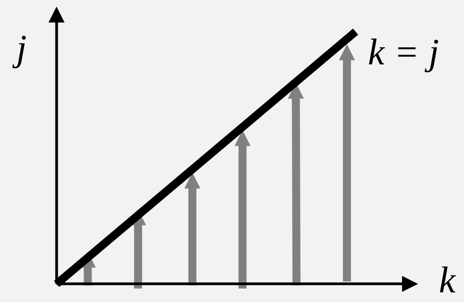
where $\mathcal{Z}\left[g(k)\right] = G(z)$ and $\mathcal{Z}\left[u(k)\right] = U(z)$

Properties of the z-Transform

Convolution

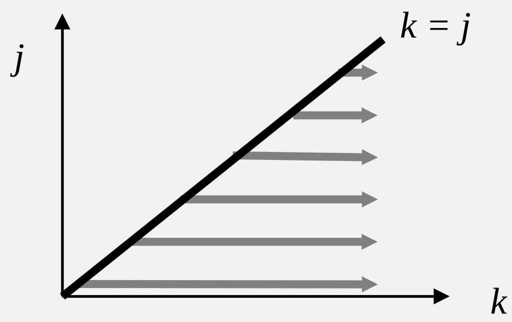
Proof:

$$\mathcal{Z}[g\otimes u(k)]=\mathcal{Z}\left[\sum_{j=0}^{k} g(k-j)\cdot u(j)\right]=\sum_{k=0}^{\infty}\left\{\left[\sum_{j=0}^{k} g(k-j)\cdot u(j)\right]\cdot z^{-k}\right\}$$



$$=\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} g(k-j)\cdot z^{-k}\cdot u(j)\right]$$

$$=\sum_{j=0}^{\infty}\left[\sum_{k=j}^{\infty} g(k-j)\cdot z^{-k}\right]\cdot u(j)$$



define $i=k-j$ $\longrightarrow$ $$=\sum_{j=0}^{\infty}\left[\sum_{i=0}^{\infty} g(i)\cdot z^{-i-j}\right]\cdot u(j)$$

$$=\left[\sum_{i=0}^{\infty} g(i)\cdot z^{-i}\right]\cdot\left[\sum_{j=0}^{\infty} u(j)\cdot z^{-j}\right]\quad=G(z)\cdot U(z)$$

Inverse z-Transform

- Given a z-transform function $X(z)$, the corresponding time domain sequence $x(k)$ can be obtained using the inverse z-transform. The inverse z-transform is defined to be

$$x(k) = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi j} \cdot \oint_C X(z) \cdot z^{k-1} dz$$

contour of integration encloses all singularity of $X(z)$

- In practice, the inverse z-transform can be obtained from:
  - Cauchy Residue Theorem
  - Direct Long Division
  - Partial Fraction Expansion
  - Computation method (e.g., impulse response)

$\mathcal{Z}^{-1}$ Using Cauchy Residue Theorem

$$x(k) = \mathcal{Z}^{-1}\left[X(z)\right] = \frac{1}{2\pi j} \cdot \oint_C X(z) \cdot z^{k-1} dz$$

the contour integration can be evaluated using the Cauchy Residue Theorem, e.g.,

$$X(z) = \frac{z}{(z-1)(z-2)}$$

$$x(k) = \frac{1}{2\pi j} \cdot 2\pi j \cdot \left(\text{sum of the residue of the integral}\right)$$

$$= \frac{1}{2\pi j} \cdot 2\pi j \cdot \left(\sum (z - p_i) \cdot X(z) \cdot z^{k-1}\Big|_{z=p_i}\right)$$

$$= \left((z-2) \cdot X(z) \cdot z^{k-1}\Big|_{z=2} + (z-1) \cdot X(z) \cdot z^{k-1}\Big|_{z=1}\right)$$

$$= \left(\frac{z}{z-1} \cdot z^{k-1}\right)_{z=2} + \left(\frac{z}{z-2} \cdot z^{k-1}\right)_{z=1} = 2^k - 1$$

$\mathcal{Z}^{-1}$ Using *Long Division*

$$X(z) = \frac{z^2 + z}{z^2 - 3z + 4} = \frac{1 + z^{-1}}{1 - 3z^{-1} + 4z^{-2}}$$

$$
\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\
1 - 3z^{-1} + 4z^{-2} & 1 + z^{-1} \\
 & \underline{1 - 3z^{-1} + 4z^{-2}} \\
 & 4z^{-1} - 4z^{-2} \\
 & \underline{4z^{-1} - 12z^{-2} + 16z^{-3}} \\
 & 8z^{-2} - 16z^{-3} \\
 & \underline{8z^{-2} - 24z^{-3} + 32z^{-4}} \\
 & 8z^{-3} - 32z^{-4} \\
 & \vdots
\end{array}
$$

$$\Rightarrow \quad x(0) = 1, \quad x(1) = 4, \quad x(2) = 8, \quad x(3) = 8, \cdots$$

Main problem: no closed-form solution

Partial Fraction Expansion

- The procedure is very similar to the one used in solving the inverse Laplace transform.

$$X(z) = \frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

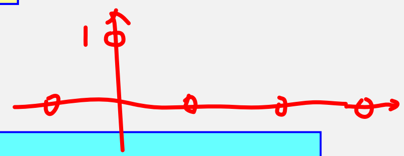
$$= A_0 + A_1\frac{z}{z-p_1} + A_2\frac{z}{z-p_2} + \cdots + A_n\frac{z}{z-p_n}$$

where

$$A_0 = X(0)$$

$$A_i = \frac{z-p_i}{z}X(z)\Big|_{z=p_i}, \quad i=1,\ 2,\ 3,\cdots$$

$\delta_0(k) =$



$\Rightarrow$

$$x(k) = \mathcal{Z}^{-1}\left[X(z)\right]$$

$$= A_0\cdot\delta_0(k) + A_1\cdot(p_1)^k + A_2\cdot(p_2)^k + \cdots + A_n\cdot(p_n)^k$$

$\mathcal{Z}^{-1}$ Using Partial Fraction Expansion

$$X(z)=\frac{0.5(1-e^{-T})^2(z^2+e^{-T}z)}{(z-1)(z-e^{-T})(z-e^{-2T})}=A_1\frac{z}{z-1}+A_2\frac{z}{z-e^{-T}}+A_3\frac{z}{z-e^{-2T}}$$

$$A_1=\frac{z-1}{z}X(z)\bigg|_{z=1}=\frac{0.5(1-e^{-T})^2(1+e^{-T})}{(1-e^{-T})(1-e^{-2T})}=0.5\frac{(1-e^{-T})(1+e^{-T})}{(1-e^{-2T})}=0.5$$

$$A_2=\frac{z-e^{-T}}{z}X(z)\bigg|_{z=e^{-T}}=-\frac{0.5(1-e^{-T})(e^{-2T}+e^{-2T})}{e^{-T}(e^{-T}-e^{-2T})}=-1$$

$$A_3=\frac{z-e^{-2T}}{z}X(z)\bigg|_{z=e^{-2T}}=\frac{0.5(1-e^{-T})^2(e^{-4T}+e^{-3T})}{e^{-2T}(e^{-2T}-1)(e^{-2T}-e^{-T})}=0.5$$

$\Rightarrow$

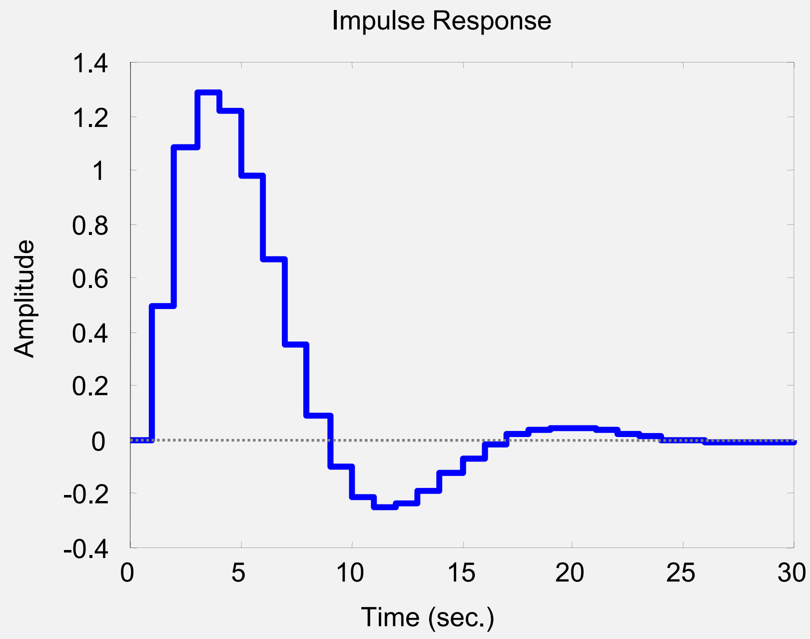
$$x(k)=\mathcal{Z}^{-1}\left[X(z)\right]=0.5-\left(e^{-T}\right)^k+0.5\left(e^{-2T}\right)^k=0.5-e^{-kT}+0.5e^{-2kT}$$

$\mathcal{Z}^{-1}$ Using Computation

$$G(z)=\frac{0.5z^{-1}+0.33z^{-2}}{1-1.5z^{-1}+0.66z^{-2}}=\frac{0.5z+0.33}{z^{2}-1.5z+0.66}$$

```
num = [0.5 0.33];
den = [1 -1.5 0.66];
g = dimpulse(num,den);
```

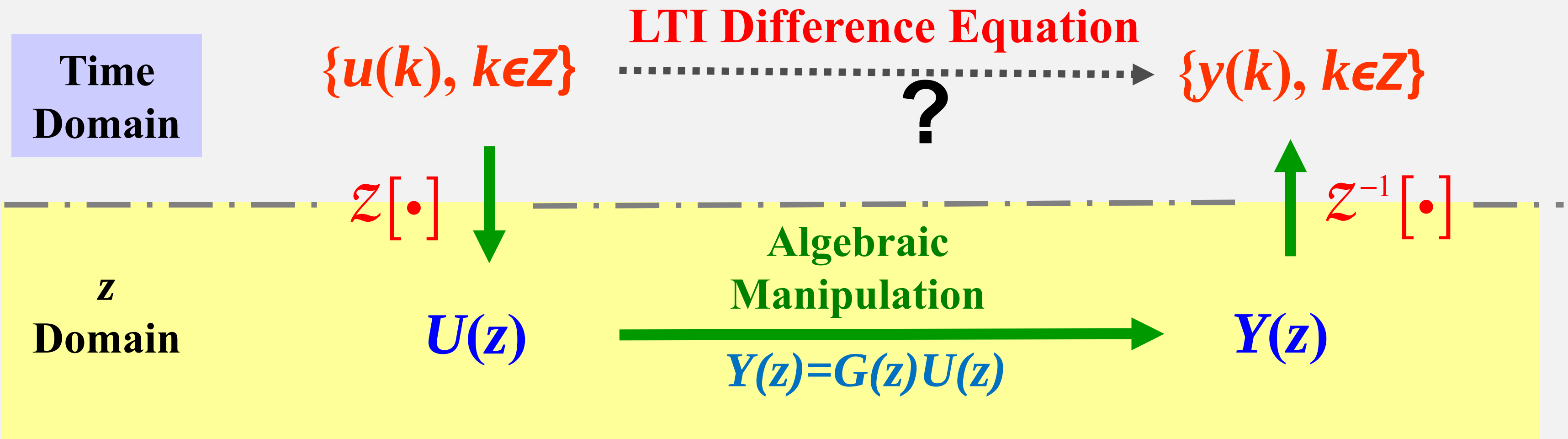
```
g =
      0
 0.5000
 1.0800
 1.2900
 1.2222
 0.9819
 0.6662
 0.3512
 0.0872
-0.1011
-0.2091
-0.2470
-0.2325
...
```



Solving Linear Difference Equations

- Linear-Time-Invariant (LTI) Difference Equation

$$y(k)+a_1\cdot y(k-1)+a_2\cdot y(k-2)+\cdots+a_n\cdot y(k-n)$$
$$=b_0\cdot u(k)+b_1\cdot u(k-1)+b_2\cdot u(k-2)+\cdots+b_n\cdot u(k-n)$$



Make use of the time shift property of the z transform:

$$\mathcal{Z}\left[x(k-d)\right]=z^{-d}\cdot X(z)$$
$$\mathcal{Z}\left[x(k+d)\right]=z^{d}X(z)-z^{d}x(0)-z^{d-1}x(1)-z^{d-2}x(2)-\cdots-z\cdot x(d-1)$$

Solving Difference Equation

Free response:

$$x(k+2)+3x(k+1)+2x(k)=0,\quad x(0)=0,\quad x(1)=1$$

$$\mathcal{Z}[x(k-d)] = z^{-d}\cdot X(z)$$

$$\mathcal{Z}[x(k+d)] = z^{d}X(z)-z^{d}x(0)-z^{d-1}x(1)-z^{d-2}x(2)-\cdots-z\cdot x(d-1)$$

$$\Rightarrow$$

$$z^2X(z)-z^2x(0)-z\cdot x(1)+3z\cdot X(z)-3z\cdot x(0)+2X(z)=0$$

$$\Rightarrow\quad X(z)=\frac{z}{z^2+3z+2}=\frac{z}{(z+1)(z+2)}=\frac{z}{z+1}-\frac{z}{z+2}$$

$f(k), k\ge 0$	$F(z)$
a^k	$\frac{z}{z-a}$

$$\Rightarrow$$

$$x(k)=\mathcal{Z}^{-1}[X(z)]=\mathcal{Z}^{-1}\left[\frac{z}{z-(-1)}\right]-\mathcal{Z}^{-1}\left[\frac{z}{z-(-2)}\right]$$

$$=(-1)^k-(-2)^k \qquad k=0,1,2,\ldots$$

Solving Difference Equation

Forced response

$$x(k+2) + 0.4 \cdot x(k+1) - 0.32 \cdot x(k) = u(k) \qquad u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

where $x(0) = 0$ and $x(1) = 1$ and $u(k)$ is a unit step input

$\Rightarrow$

$$z^2 X(z) - z^2 x(0) - z \cdot x(1) + 0.4 \cdot z \cdot X(z) - 0.4 \cdot z \cdot x(0)$$

$$\Rightarrow \qquad -0.32 \cdot X(z) = \frac{z}{z-1}$$

$$X(z) = \frac{z^2}{(z-1)(z^2+0.4z-0.32)} = \frac{z^2}{(z-1)(z+0.8)(z-0.4)}$$

$$\Rightarrow \quad X(z) = 0.926\frac{z}{z-1} - 0.3704\frac{z}{z+0.8} - 0.5556\frac{z}{z-0.4}$$

$$\Rightarrow \quad x(k) = 0.926 - 0.3704 \cdot (-0.8)^k - 0.5556 \cdot (0.4)^k$$

Pulse Transfer Function

- The transfer function for the continuous-time system relates the Laplace transform of the continuous-time output to that of the continuous-time input described by LTI differential equations.

- For discrete-time systems, the pulse transfer function relates the *z*-transform of the output sequence (e.g., the calculated control input values at the sample instances by microprocessor) to that of the input sequence (e.g., the representation of a sampled signal in discrete-time domain) described by LTI difference equations.

Pulse Transfer Function

Consider a discrete system with ***zero initial conditions*** described by the following LTI difference equation

$$y(k)+a_1\cdot y(k-1)+a_2\cdot y(k-2)=b_0\cdot u(k)+b_1\cdot u(k-1)+b_2\cdot u(k-2)$$

Take z transform:

$$Y(z)+a_1\cdot z^{-1}Y(z)+a_2\cdot z^{-2}Y(z)=b_0\cdot U(z)+b_1\cdot z^{-1}U(z)+b_2\cdot z^{-2}U(z)$$

Group Terms:

$$\left(1+a_1\cdot z^{-1}+a_2\cdot z^{-2}\right)\cdot Y(z)=\left(b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}\right)\cdot U(z)$$

$$\Rightarrow\quad Y(z)=\left(\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}}{1+a_1\cdot z^{-1}+a_2\cdot z^{-2}}\right)\cdot U(z)=G(z)\cdot U(z)$$

Pulse Transfer Function:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}}{1+a_1\cdot z^{-1}+a_2\cdot z^{-2}}\left(=\frac{b_0\cdot z^2+b_1\cdot z^1+b_2}{z^2+a_1\cdot z^1+a_2}=\frac{N(z)}{D(z)}\right)$$

Pulse Transfer Function

Observation: both the following two LTI difference equations have the same pulse transfer function:

$$\begin{cases} y(k)+a_1\cdot y(k-1)+a_2\cdot y(k-2)=b_0\cdot u(k)+b_1\cdot u(k-1)+b_2\cdot u(k-2) \\ y(k+2)+a_1\cdot y(k+1)+a_2\cdot y(k)=b_0\cdot u(k+2)+b_1\cdot u(k+1)+b_2\cdot u(k) \end{cases}$$

- In general, given an *n*th order difference equation

$$y(k)+\boldsymbol{a_1}\cdot y(k-1)+\boldsymbol{a_2}\cdot y(k-2)+\cdots+\boldsymbol{a_n}\cdot y(k-n)$$
$$=\boldsymbol{b_0}\cdot u(k)+\boldsymbol{b_1}\cdot u(k-1)+\boldsymbol{b_2}\cdot u(k-2)+\cdots+\boldsymbol{b_n}\cdot u(k-n)$$

The corresponding pulse transfer function is:

$$G(z)=\frac{Y(z)}{U(z)}=\frac{\boldsymbol{b_0}+\boldsymbol{b_1}\cdot z^{-1}+\boldsymbol{b_2}\cdot z^{-2}+\cdots+\boldsymbol{b_n}\cdot z^{-n}}{1+\boldsymbol{a_1}\cdot z^{-1}+\boldsymbol{a_2}\cdot z^{-2}+\cdots+\boldsymbol{a_n}\cdot z^{-n}}\left(=\frac{N(z)}{D(z)}\right)$$

Pulse Transfer Function

- **Realizable** (***causal***)

  A pulse transfer function is realizable if the highest order of the denominator polynomial is larger than or equal to that of the numerator polynomial, i.e. $O[D(z)] \geq O[N(z)]$

  Let:

$$G_1(z) = \frac{4z+5}{z^2+3z+4}, \qquad \text{(Realizable)}$$

$$G_2\left(z^{-1}\right) = \frac{4+5z^{-1}}{1+3z^{-1}+4z^{-2}}, \quad G_3\left(z^{-1}\right) = \frac{4z^{-2}+5z^{-3}}{1+3z^{-1}+4z^{-2}}, \quad G_4(z) = \frac{4+5z^{-1}}{z^{-1}+3z^{-2}+4z^{-3}}$$

$$\Rightarrow \quad G_2\left(z^{-1}\right) = z \cdot G_1(z) \quad \rightarrow \quad \text{Realizable}$$

$$\Rightarrow \quad G_3\left(z^{-1}\right) = z^{-1} \cdot G_1(z) \quad \rightarrow \quad \text{Realizable}$$

$$\Rightarrow \quad G_4(z) = z^2 \cdot G_1(z) \quad \rightarrow \quad \text{Not Realizable}$$

It is preferable to represent the transfer function as rational functions of z when concerning with the order/pole/zero of the transfer function

Pulse Transfer Function

- ***Why the name***?

  Recall for continuous-time systems

$$Y(s) = G(s) \cdot U(s)$$

  If $U(s) = 1$, i.e. unit impulse input,

$$Y(s) = G(s) \cdot U(s) = G(s) \quad \Rightarrow \quad y_I(t) = g(t) = \mathcal{L}^{-1}\left[G(s)\right]$$

  Hence, $G(s)$ is the $\mathcal{L}$-transform of the impulse response of the system!

Define: Unit Pulse function for discrete-time systems

$$u(k) = \delta_0(k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases} \quad \Rightarrow \quad U(z) = \mathcal{Z}\left[u(k)\right] = \mathcal{Z}\left[\delta_0(k)\right] = 1$$

Therefore:

$$Y(z) = G(z) \cdot U(z) \quad \Rightarrow \quad y_I(k) = g(k) = \mathcal{Z}^{-1}\left[G(z)\right]$$

G(z) is the z-transform of the response of the discrete-time system under unit pulse input. In other word, g(k) is the unit pulse response.

Pulse Transfer Function

Since any input sequence $\{u(k), k \in Z\}$ can be represented by

$$u = u(0)\cdot\delta_0(k) + u(1)\cdot\delta_0(k-1) + u(2)\cdot\delta_0(k-2) + \cdots$$

$$= \sum_{j=0}^{\infty} u(j)\cdot\delta_0(k-j)$$

Then

$$y(k) = u(0)\cdot \mathbf{g(k)} + u(1)\cdot \mathbf{g(k-1)} + u(2)\cdot \mathbf{g(k-2)} + \cdots \qquad \textbf{\textit{Why?}}$$

$$\Rightarrow \quad y(k) = \sum_{j=0}^{k} u(j)\cdot g(k-j) = g \otimes u(k)$$

The response of an LTI discrete-time system to any input sequence u(k) can be calculated by convolving the system unit pulse response sequence g(k) with the input sequence.

Take z-transform:

$$Y(z) = \mathcal{Z}\left[g \otimes u(k)\right] = \mathcal{Z}\left[g(k)\right]\cdot\mathcal{Z}\left[u(k)\right] = G(z)\cdot U(z)$$

Finite Impulse Response (FIR) System

Consider the following system:

$$y(k+3) = 2 \cdot u(k+3) - u(k+2) + 4 \cdot u(k+1) + u(k)$$

$$\Rightarrow \quad G(z) = \frac{2z^3 - 1z^2 + 4z + 1}{z^3} = 2 + (-1) \cdot z^{-1} + 4 \cdot z^{-2} + 1 \cdot z^{-3}$$

$$\Rightarrow \quad g(k) = \mathcal{Z}^{-1}\left[G(z)\right] = \mathcal{Z}^{-1}\left[2 - z^{-1} + 4z^{-2} + z^{-3}\right]$$

$$= 2 \cdot \delta_0(k) - \delta_0(k-1) + 4 \cdot \delta_0(k-2) + \delta_0(k-3)$$

$$\Rightarrow \quad g(0) = 2, \; g(1) = -1, \; g(2) = 4, \; g(3) = 1, \quad g(k) = 0, \quad \forall\, k > 3$$

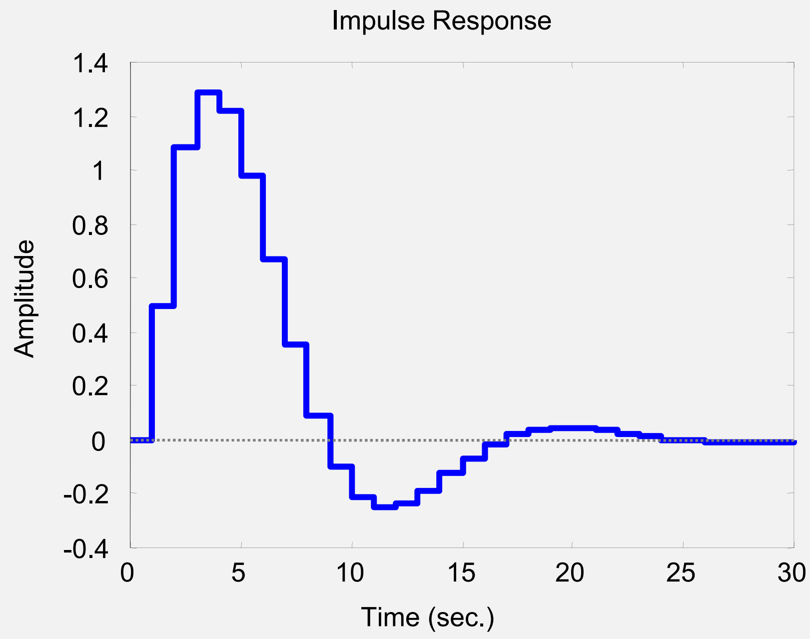
The pulse response of a system with *all of its poles at the origin* will have *finite non-zero terms*. Pulse response of this type is called finite impulse response (FIR) and the system (digital filter) that have all its poles at the origin is called a *finite impulse response* (FIR) *filter*.

Infinite Impulse Response (IIR)

$$G(z) = \frac{0.5z^{-1} + 0.33z^{-2}}{1 - 1.5z^{-1} + 0.66z^{-2}} = \frac{0.5z + 0.33}{z^2 - 1.5z + 0.66}$$

```
num = [0.5 0.33];
den = [1 -1.5 0.66];
g = dimpulse(num,den);
```

```
g =
        0
   0.5000
   1.0800
   1.2900
   1.2222
   0.9819
   0.6662
   0.3512
   0.0872
  -0.1011
  -0.2091
  -0.2470
  -0.2325
...
```



Frequency Response

- *Steady-state* responses of a *stable* system under sinusoidal inputs. Given a discrete-time system

$$G(z)=\frac{Y(z)}{U(z)}=\frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

where $p_i \in C$, and $|p_i| < 1$ for all i. Let the input to the system be a cosine sequence with frequency ω, i.e.

$$u(k)=A\cdot\cos(\omega kT)=\frac{A}{2}\left(e^{j\omega kT}+e^{-j\omega kT}\right)$$

$$\Rightarrow \quad U(z)=\frac{A}{2}\left(\frac{z}{z-e^{j\omega T}}+\frac{z}{z-e^{-j\omega T}}\right)$$

Then,

$$Y(z)=G(z)\cdot U(z)=\frac{N(z)}{(z-p_1)(z-p_2)\cdots(z-p_n)}\cdot\frac{A}{2}\left(\frac{z}{z-e^{j\omega T}}+\frac{z}{z-e^{-j\omega T}}\right)$$

Frequency Response

Perform partial fraction expansion

$$\Rightarrow \quad Y(z) = B\frac{z}{z-e^{j\omega T}} + C\frac{z}{z-e^{-j\omega T}} + \sum_{i=1}^{n} D_i \frac{z}{z-p_i}$$

$\rightarrow 0$, since system is stable

terms due to poles of $G(z)$

$$B = \frac{z-e^{j\omega T}}{z}Y(z)\bigg|_{z=e^{j\omega T}} = \frac{A}{2}\left[1+\frac{z-e^{j\omega T}}{z-e^{-j\omega T}}\right]G(z)\bigg|_{z=e^{j\omega T}} = \frac{A}{2}G\left(e^{j\omega T}\right)$$

$$C = \frac{z-e^{-j\omega T}}{z}Y(z)\bigg|_{z=e^{-j\omega T}} = \frac{A}{2}\left[\frac{z-e^{-j\omega T}}{z-e^{j\omega T}}+1\right]G(z)\bigg|_{z=e^{-j\omega T}} = \frac{A}{2}G\left(e^{-j\omega T}\right)$$

At steady state

$$\Rightarrow \quad Y_{SS}(z) = \frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z-e^{j\omega T}} + G\left(e^{-j\omega T}\right)\frac{z}{z-e^{-j\omega T}}\right]$$

Frequency Response

Let

$$G\left(e^{j\omega T}\right)=\left|G\left(e^{j\omega T}\right)\right|\cdot e^{j\angle G\left(e^{j\omega T}\right)}=\left|G\left(e^{j\omega T}\right)\right|\cdot e^{j\phi},\quad \phi=\angle G\left(e^{j\omega T}\right)$$

Then

$$G\left(e^{-j\omega T}\right)=\left|G\left(e^{-j\omega T}\right)\right|\cdot e^{j\angle G\left(e^{-j\omega T}\right)}=\left|G\left(e^{j\omega T}\right)\right|\cdot e^{-j\phi}$$

Hence

$$Y_{SS}(z)=\frac{A}{2}\left[G\left(e^{j\omega T}\right)\frac{z}{z-e^{j\omega T}}+G\left(e^{-j\omega T}\right)\frac{z}{z-e^{-j\omega T}}\right]$$

$$=\frac{A}{2}\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\left[e^{j\phi}\frac{z}{z-e^{j\omega T}}+e^{-j\phi}\frac{z}{z-e^{-j\omega T}}\right]$$

Frequency Response

$$Y_{SS}(z) = \frac{A}{2}\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\left[e^{j\phi}\frac{z}{z-e^{j\omega T}} + e^{-j\phi}\frac{z}{z-e^{-j\omega T}}\right]$$

Take inverse *z*-transform

$$y_{SS}(k) = \frac{A}{2}\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\left[e^{j\phi}\left(e^{j\omega T}\right)^k + e^{-j\phi}\left(e^{-j\omega T}\right)^k\right]$$

$$= A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\frac{1}{2}\left(e^{j(\omega kT+\phi)} + e^{-j(\omega kT+\phi)}\right)$$

$\Rightarrow$

$$y_{SS}(k) = A\cdot\left|G\left(e^{j\omega T}\right)\right|\cdot\cos\left(\omega kT+\phi\right)$$

where $\phi = \angle G\left(e^{j\omega T}\right)$

Frequency Response

$$y_{SS}(k) = A \cdot \left| G\left(e^{j\omega T}\right) \right| \cdot \cos\left(\omega kT + \phi\right) \leftarrow \boxed{G(z)} \leftarrow u(k) = A \cdot \cos\left(\omega kT\right)$$

- Similar to the continuous-time case, the steady-state response of the system $G(z)$ to a sinusoidal input of frequency ω is still sinusoidal with the same frequency but scaled in amplitude and shifted in phase
- The amplitude of the steady-state response is scaled by a factor $|G(e^{j\omega T})|$, which is referred to as the system gain at frequency ω
- The phase of the response is shifted in time by $\angle G(e^{j\omega T})$, which is referred to as the phase of the system at frequency ω
- The frequency response function of a discrete system $G(z)$ can be obtained by replacing the z-transform complex variable z with e^{jzT}, i.e.

$$G\left(e^{j\omega T}\right) = G(z)\Big|_{z=e^{j\omega T}} = G\left(\cos\left(\omega T\right) + j\sin\left(\omega T\right)\right)$$

Frequency Response

- Steady State Gain (DC gain)
  The steady state gain of a discrete-time system can be obtained by letting $\omega = 0$, i.e.

$$\text{DC Gain} = G\left(e^{j\omega T}\right)\Big|_{\omega=0} = G(z)\big|_{z=1} = G(1)$$

- Periodic Frequency Response
  Since

$$G\left(e^{j(\omega \pm N\omega_S)T}\right) = G\left[\cos\left((\omega \pm N\omega_S)T\right) + j\sin\left((\omega \pm N\omega_S)T\right)\right]$$

$$= G\left[\cos(\omega T) + j\sin(\omega T)\right]$$

$$= G\left(e^{j\omega T}\right), \qquad \forall\, N$$

$\Rightarrow$

D.T. system frequency response is periodic with period $\omega_S = \dfrac{2\pi}{T}$

Frequency Response

- **Example**

Given a discrete-time system

$$y(k) = e^{-2T} \cdot y(k-1) + u(k),$$

where $T = \pi/5$, i.e., $\omega_S = 2\pi/T = 10$ rad/sec

Take z-transform

$$Y(z) = e^{-2T} \cdot z^{-1} Y(z) + U(z) \qquad \Rightarrow \qquad G(z) = \frac{z}{z - e^{-2T}}$$

$$\Rightarrow \quad G(e^{j\omega T}) = \frac{e^{j\omega T}}{e^{j\omega T} - e^{-2T}}$$

$$\left|G(e^{j\omega T})\right| = \frac{\left|e^{j\omega T}\right|}{\left|e^{j\omega T} - e^{-2T}\right|} = \frac{1}{\sqrt{\left(\cos(\omega T) - e^{-2T}\right)^2 + \sin^2(\omega T)}}$$

$$\angle G(e^{j\omega T}) = \angle\left(e^{j\omega T}\right) - \angle\left(e^{j\omega T} - e^{-2T}\right)$$
$$= \omega T - \operatorname{atan2}\left(\sin(\omega T), \cos(\omega T) - e^{-2T}\right)$$

Frequency Response

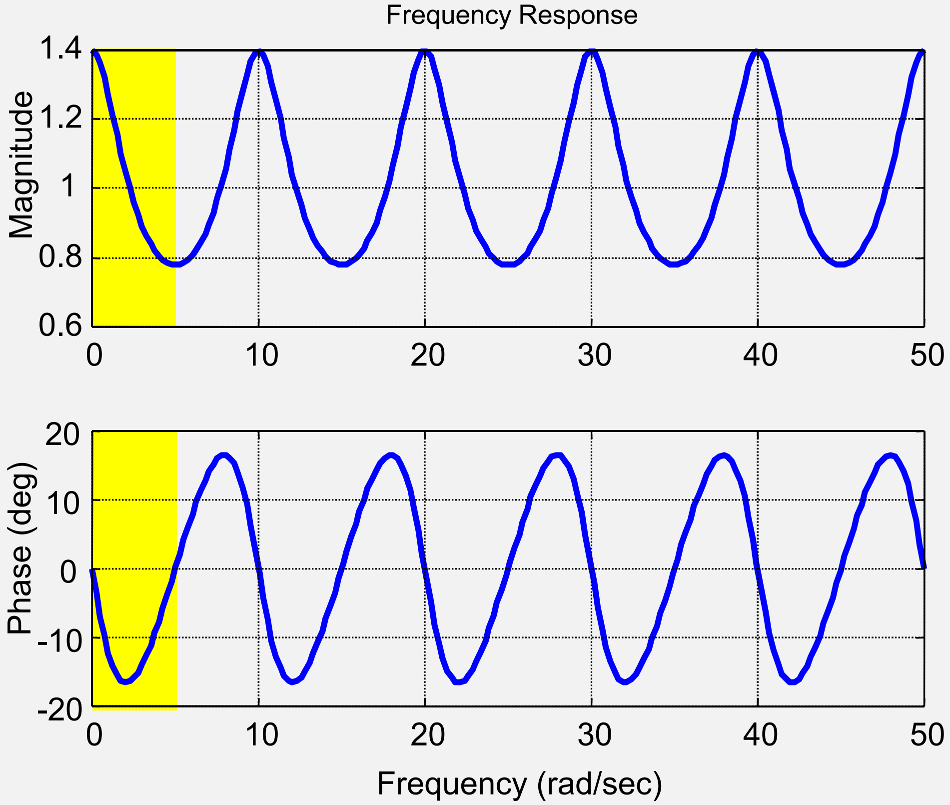
$$G(z) = \frac{z}{z - e^{-2T}} \quad \Rightarrow \quad G\left(e^{j\omega T}\right) = \frac{e^{j\omega T}}{e^{j\omega T} - e^{-2T}}$$

```
T = pi/5;
G = tf([1 0],[1 -exp(-2*T)],T);

% Set up frequency vector:
w = linspace(0,50,200);
out = freqresp(G,w);

for i = 1:length(w)
    fr(i,1) = out(:,:,i);
end

subplot(211);plot(w,abs(fr));
subplot(212);
plot(w,180/pi*angle(fr));
```

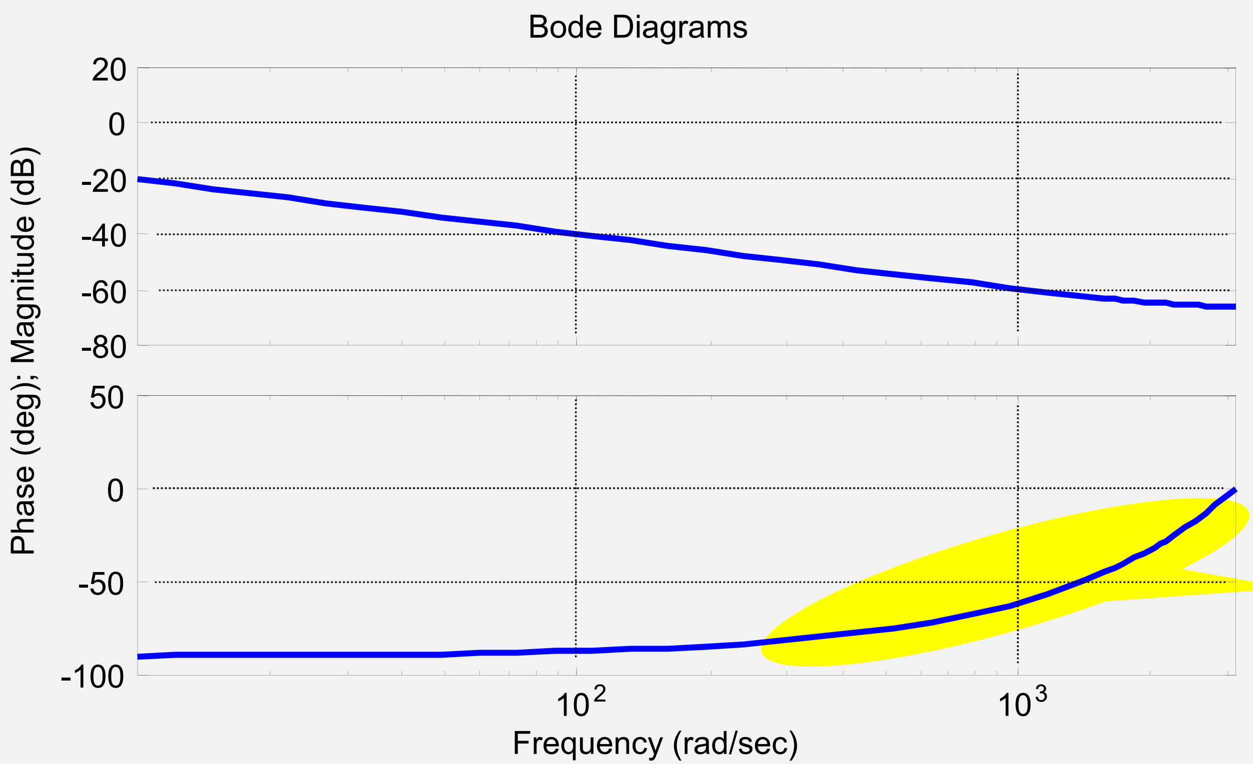


Frequency Response

- **Example** – Integrator

$$G(s)=\frac{1}{s} \quad \rightarrow \quad G(z)=\frac{Tz}{z-1} \quad \Rightarrow \quad G\left(e^{j\omega T}\right)=\frac{T\cdot e^{j\omega T}}{e^{j\omega T}-1}$$

```
» dbode([0.001 0], [1 -1], 0.001);  % sampling frequency = 1 kHz
```



Phase distortion due to aliasing

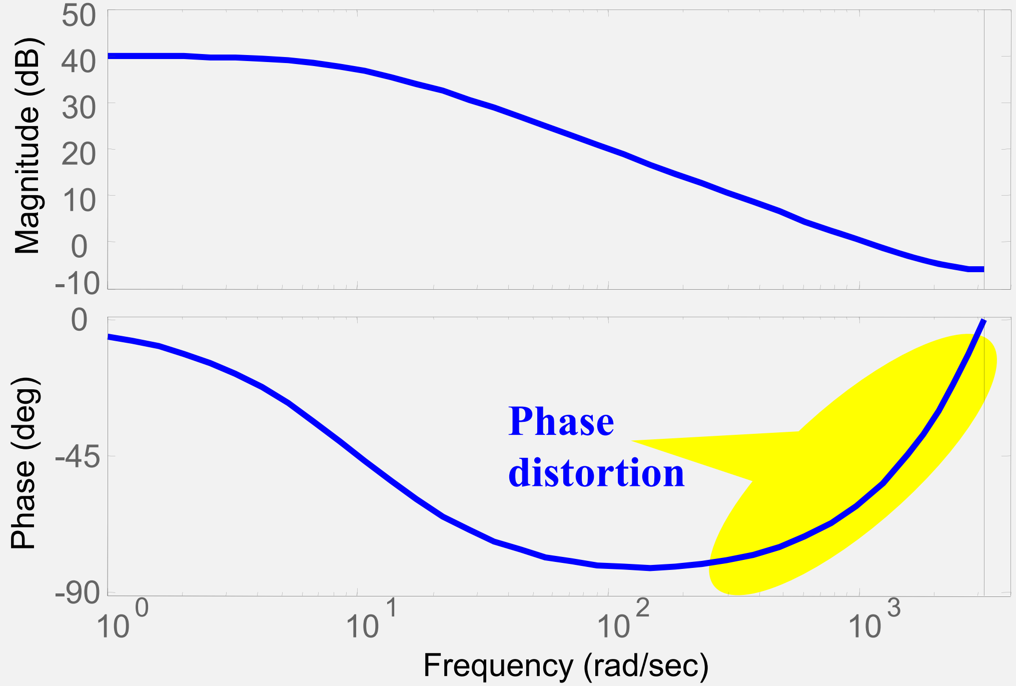
Frequency Response

- **Example** – Simple pole

$$G(s)=\frac{1}{s+a}\;\rightarrow\; g(t)=e^{-at}\;\rightarrow\; g(kT)=e^{-akT}$$

$$\rightarrow G(z)=\frac{1}{1-e^{-aT}z^{-1}}\quad\Rightarrow\quad G\left(e^{j\omega T}\right)=\frac{1}{1-e^{-aT}e^{j\omega T}}$$

```
>> a = 10; T = 0.001;
>> dbode([1 0],[1 -exp(-a*T)],T)
```



Frequency Response

- **Example** – Harmonic Oscillator

$$G(s) = \frac{\omega}{s^2 + \omega^2} \rightarrow g(t) = \sin(\omega t) \rightarrow g(kT) = \sin(\omega kT)$$

$$\rightarrow G(z) = \frac{z^{-1}\sin(\omega T)}{1 - 2z^{-1}\cos(\omega T) + z^{-2}} \Rightarrow G\left(e^{j\omega T}\right) = \frac{e^{-j\omega T}\sin(\omega T)}{1 - 2e^{-j\omega T}\cos(\omega T) + e^{-2j\omega T}}$$