Standard Forms for System Models

- **State Space Model Representation**
  - Basic concepts
  - Example
  - General form
- **Input/Output Model Representation**
  - General Form
  - Example
- **Comments on State Space and Input/Output Model Representations**

Basic Concepts of State Space Model

State Variables

- **State**

  The smallest set of variables $\{q_1, q_2, \ldots, q_n\}$ such that the knowledge of these variables at time $t = t_0$, together with the knowledge of the input for $t \geq t_0$ completely determines the behavior (the values of the state variables) of the system for time $t \geq t_0$.

- **State Vector**

  A concise mathematical representation of ALL state variables $\{q_1, q_2, \ldots, q_n\}$ in a vector form.

$$x = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix} \in R^n$$

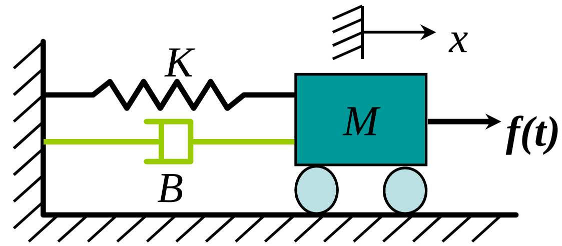
- **State Space**

  A space whose coordinates consist of state variables is called a state space. Any state can be represented by a point in state space.

One Example

EOM:

$$M\ddot{x} + B\dot{x} + Kx = f(t)$$



Q: What information about the mass do we need to know to be able to solve for x(t) for $t \geq t_0$?

Input: $\{ f(t),\ t \geq 0 \}$

$$\underset{\sim}{x} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Initial Conditions (ICs):

$x(0)$, $\dot{x}(0)$

$q_1 = x$

$q_2 = \dot{x}$

Rule of Thumb

Number of state variables = Sum of orders of EOMs

State Space Model Representation

- **Two parts:**
  - A set of first order ODEs that represents the derivative of each state variable q_i as an algebraic function of the set of state variables $\{q_i\}$ and the inputs $\{u_i\}$ only.

$$\begin{cases} \dot{q}_1 = f_1\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \\ \dot{q}_2 = f_2\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \\ \quad\vdots \\ \dot{q}_n = f_n\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \end{cases}$$

  - A set of equations that represents the output variables as algebraic functions of the set of state variables $\{q_i\}$ and the inputs $\{u_i\}$ only.

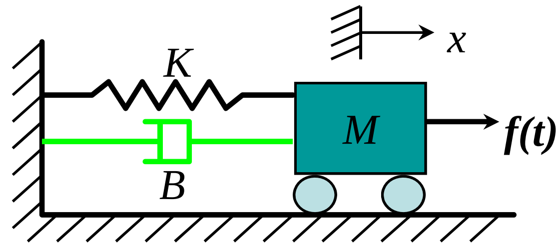
$$\begin{cases} y_1 = g_1\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \\ y_2 = g_2\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \\ \quad\vdots \\ y_p = g_p\left(t, q_1, q_2, \ldots, q_n, u_1, \ldots, u_m\right) \end{cases}$$

State Space Model Representation

- Example

EOM

$$M\ddot{x} + B\dot{x} + Kx = f(t)$$



State Variables:

$$q_1 = x\,, \qquad q_2 = \dot{x}\,, \qquad \mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Outputs: ***(assuming that we are interested in motion and damper force)***

$$y_1 = x\,, \qquad y_2 = B\dot{x}$$

State Space Representation:

State equation

$$\dot{q}_1 = \dot{x} = q_2$$

$$\dot{q}_2 = \ddot{x} = \frac{1}{M}\left\{ f - B\dot{x} - Kx \right\}$$

$$= -\frac{K}{M}q_1 - \frac{B}{M}q_2 + \frac{1}{M}f$$

Output equation

$$y_1 = q_1$$

$$y_2 = Bq_2$$

Matrix Form

$$\underbrace{\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}}_{\dot{\mathbf{x}}} = \overbrace{\begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & -\frac{B}{M} \end{bmatrix}}^{A} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}}^{B} f$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \overbrace{\begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}}^{C} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{\mathbf{x}} + \overbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}^{D} f$$

State Space Model Representation

- **Obtaining State Space Representation**
  - Identify State Variables
    - *Rule of Thumb:*
      - *N-th order ODE requires N state variables.*
      - *Position and velocity of inertia elements are natural state variables for translational mechanical systems.*
  - Eliminate all algebraic equations written in the modeling process.
  - Express the resulting differential equations in terms of state variables and inputs in coupled first order ODEs.
  - Express the output variables as algebraic functions of the state variables and inputs.
  - For linear systems, put the equations in matrix form.

$$\underbrace{\dot{\mathbf{x}}}_{\substack{\text{First derivative}\\ \text{of State Vector}}} = \mathbf{A}\cdot \underbrace{\mathbf{x}}_{\text{State Vector}} + \mathbf{B}\cdot \underbrace{\mathbf{u}}_{\text{Input Vector}}$$

$$\underbrace{\mathbf{y}}_{\text{Output Vector}} = \mathbf{C}\cdot\mathbf{x} + \mathbf{D}\cdot\mathbf{u}$$

PURDUE MECHANICAL ENGINEERING

$$\dot{q}_4 = \ddot{x}_2 = \frac{1}{M_2}\left[B_1\dot{x}_1 - B_1\dot{x}_2 + K_1x_1 - (K_1+K_2)x_2 + K_1x_p - M_2 g\right]$$

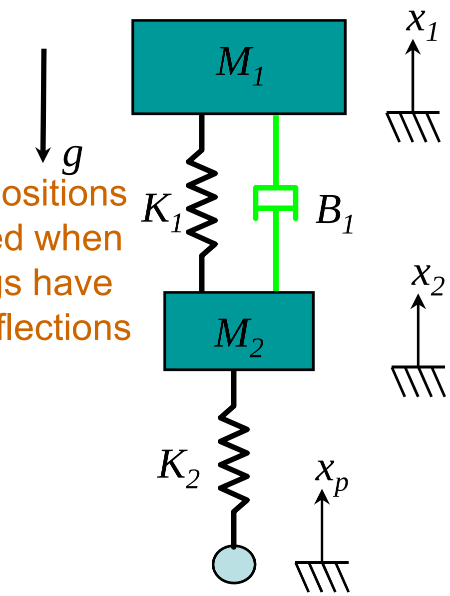
- Exercise

$$\dot{q}_2 = \ddot{x}_1 = \frac{1}{M_1}\left[-B_1\dot{x}_1 + B_1\dot{x}_2 - K_1x_1 + K_1x_2 - M_1 g\right]$$

Assuming we are interested in the car body movement (for ride quality study) and the deflection of suspension (spring K_1) (for implementation).

$$M_1\ddot{x}_1 + B_1\dot{x}_1 - B_1\dot{x}_2 + K_1x_1 - K_1x_2 = -M_1 g$$

$$M_2\ddot{x}_2 - B_1\dot{x}_1 + B_1\dot{x}_2 - K_1x_1 + (K_1 + K_2)x_2 = K_2x_p - M_2 g$$



Ref. positions defined when springs have no deflections

State Variables:

$q_1 = x_1, \quad q_2 = \dot{x}_1, \quad q_3 = x_2, \quad q_4 = \dot{x}_2$

Outputs:

$y_1 = x_1, \quad y_2 = x_1 - x_2$

Inputs:

$x_p, \quad M_1 g, \quad M_2 g$

State Space Representation:

$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{K_1}{M_1} & -\frac{B_1}{M_1} & \frac{K_1}{M_1} & \frac{B_1}{M_1} \\ 0 & 0 & 0 & 1 \\ \frac{K_1}{M_2} & \frac{B_1}{M_2} & -\frac{K_1+K_2}{M_2} & -\frac{B_1}{M_2} \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{M_1} & 0 \\ 0 & 0 & 0 \\ \frac{K_1}{M_1} & 0 & -\frac{1}{M_1} \end{bmatrix}\begin{bmatrix} x_p \\ M_1 g \\ M_2 g \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_p \\ M_1 g \\ M_2 g \end{bmatrix}$$

Input/Output Representation

- Input/Output Model

  Uses higher order ODEs to relate the output variables, y(t), to the input variables, u(t), of a system directly.

  For single-output (SO) linear time-invariant (LTI) systems with two inputs, it can be represented by :

$$y^{(n)} + \cdots + a_2\ddot{y} + a_1\dot{y} + a_0 y = b_{m_1} u_1^{(m_1)} + \cdots + b_1\dot{u}_1 + b_0 u_1 + c_{m_2} u_2^{(m_2)} + \cdots + c_1\dot{u}_2 + c_0 u_2$$

  where

$$y^{(n)} = \left(d/dt\right)^n y$$

  – To solve an input/output differential equation, we need to know

    Inputs: (1) all inputs $u_i(t)$, $t \geq 0$

    Initial Conditions (ICs):

    (2) I.C.s: $y(0)$, $\dot{y}(0)$, $\ddot{y}(0)$, $\cdots$, $y^{(n-1)}(0)$

  – To obtain I/O models:
    - Identify input/output variables.
    - Derive equations of motion.
    - Combine equations of motion by eliminating all variables except the input and output variables and their derivatives.

PURDUE MECHANICAL ENGINEERING

Input/Output Representation

- **Example**

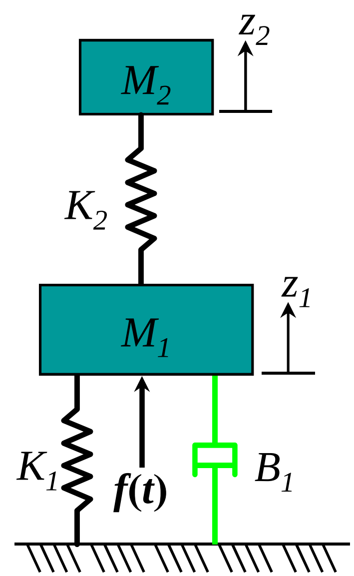
  Vibration Absorber

  EOM: $M_1\ddot{z}_1 + B_1\dot{z}_1 + (K_1 + K_2)z_1 - K_2 z_2 = f(t)$ (1)

  $M_2\ddot{z}_2 + K_2 z_2 - K_2 z_1 = 0$ (2)

  – Find input/output representation between input $f(t)$ and output z_1.

  $y = z_1, \quad u = f(t)$



  - **Need to eliminate z_2 and its time derivatives of all orders**

From (1): $z_2 = \frac{1}{K_2}\left[M_1\ddot{z}_1 + B_1\dot{z}_1 + (K_1+K_2)z_1 - f(t)\right]$

$\Rightarrow \dot{z}_2 = \frac{1}{K_2}\left[M_1\dddot{z}_1 + B_1\ddot{z}_1 + (K_1+K_2)\dot{z}_1 - \dot{f}(t)\right]$ (3)

$\Rightarrow \ddot{z}_2 = \frac{1}{K_2}\left[M_1 z_1^{(4)} + B_1\dddot{z}_1 + (K_1+K_2)\ddot{z}_1 - \ddot{f}(t)\right]$ (4)

Substitute (3), (4) into (2):

$$\frac{M_1M_2}{K_2}z_1^{(4)} + \frac{M_2B_1}{K_2}z_1^{(3)} + \frac{M_2(K_1+K_2)}{K_2}\ddot{z}_1 - \frac{M_2}{K_2}\ddot{f} + M_1\ddot{z}_1 + B_1\dot{z}_1 + (K_1+K_2)z_1 - f - K_2 z_1 = 0$$

$\Rightarrow$

$$\boxed{z_1^{(4)} + \frac{B_1}{M_1}z_1^{(3)} + \frac{M_1K_2 + M_2(K_1+K_2)}{M_1M_2}\ddot{z}_1 + \frac{B_1K_2}{M_1M_2}\dot{z}_1 + \frac{K_1K_2}{M_1M_2}z_1 = \frac{1}{M_1}\ddot{f} + \frac{K_2}{M_1M_2}f}$$

Exercise: Verify that I/O model when output is defined to be z_2 is

$$\boxed{z_2^{(4)} + \frac{B_1}{M_1}z_2^{(3)} + \frac{M_1K_2 + M_2(K_1+K_2)}{M_1M_2}\ddot{z}_2 + \frac{B_1K_2}{M_1M_2}\dot{z}_2 + \frac{K_1K_2}{M_1M_2}z_2 = \frac{K_2}{M_1M_2}f}$$

Input/Output Models vs State-Space Models

- **State Space Models:**
  - Consider the internal behavior of a system
  - Can handle complicated output variables easily
  - Have significant computation advantage for computer simulation
  - Can represent multi-inputs-multi-outputs (MIMO) systems and nonlinear systems
- **Input/Output Models:**
  - Conceptually simple
  - Easy to be converted to frequency domain transfer functions that are more intuitive to practicing engineers
  - Difficult to solve in time domain (solution: using Laplace transformation and solve in frequency domain)