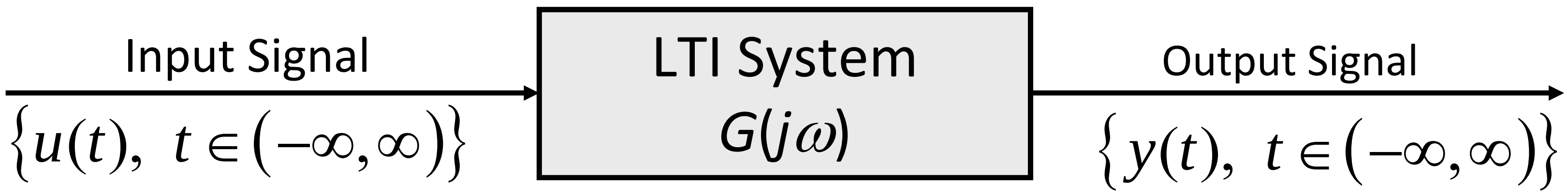
Fourier Spectral Analysis

- **Signals**
- **Periodic Signal**
  - Fourier Series Representation of Periodic Signals
- **Frequency Spectra**
  - Amplitude and Phase Spectra of Signals
  - Signals Through Systems - a Frequency Spectrum Perspective
- **Non-periodic Signals - Fourier Transform**
- **Random Signals - Power Spectral Density**

PURDUE MECHANICAL ENGINEERING

Signals



Completely characterized by its Frequency Response Function

$$\left\{ G(j\omega) = \left| G(j\omega) \right| e^{j\angle G(j\omega)},\ \ \omega \in \left[0, \infty\right) \right\}$$

- Signals can be categorized as:
  - Periodic Signals
  - Non-Periodic Signals (well defined)
  - Random Signals

$\Rightarrow$ *Would like to characterize signals in the frequency domain*!

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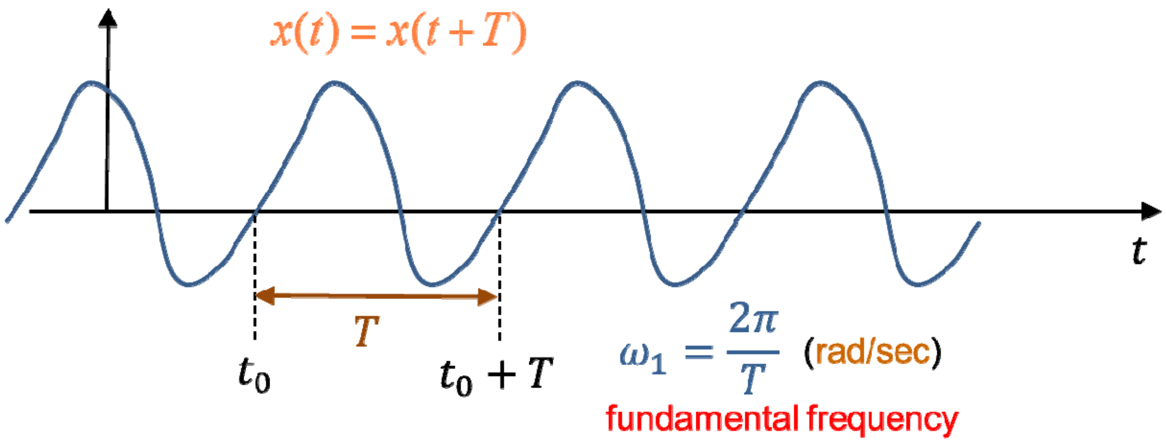
Periodic Signals - Fourier Series

Any periodic function $x(t)$, of period T, can be represented by an infinite series of sine and cosine functions of integer multiples of its fundamental frequency $\omega_1 = 2\pi/T$:



Fourier Series:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} [A_k \cos(k\omega_1 t) + B_k \sin(k\omega_1 t)]$$

Fourier Coefficients:

$$A_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos(k\omega_1 t)dt \qquad B_k = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\sin(k\omega_1 t)dt$$

$\frac{A_0}{2}$: DC component $\quad k\omega_1 = k\frac{2\pi}{T}$: kth harmonic frequency, $k \in \mathbb{Z}^+$

Periodic Signals - Fourier Series

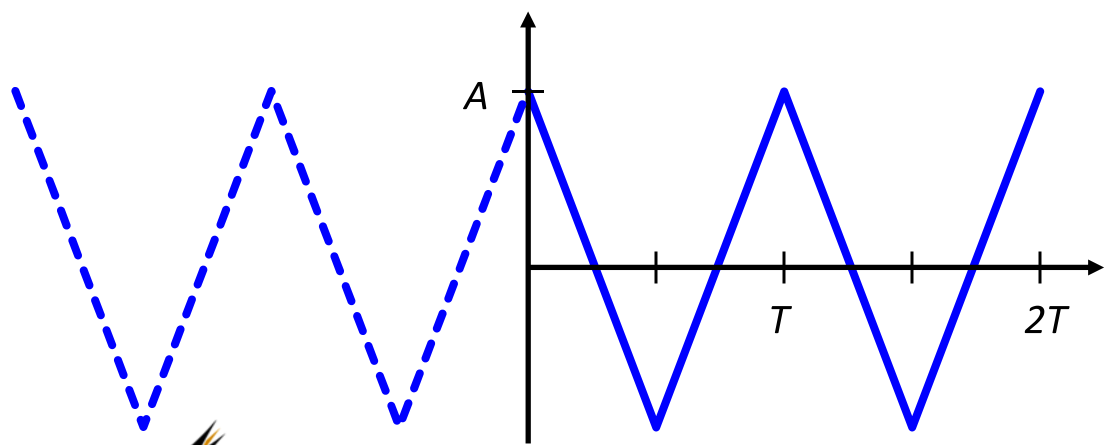
Ex: triangle signal with period T sec

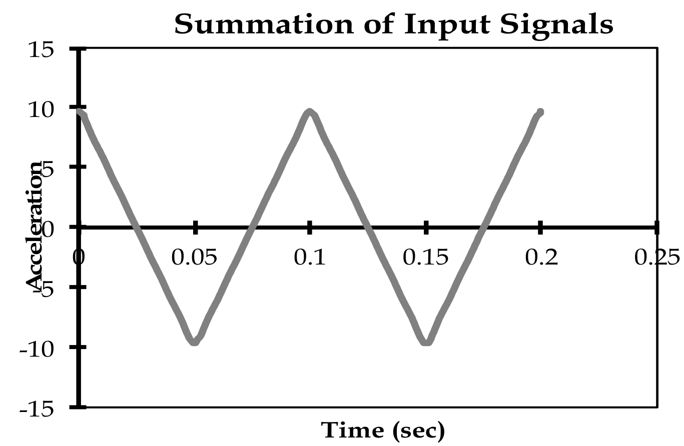
$$x(t)=\sum_{k=1}^{\infty}\underbrace{\left(\frac{8A}{\pi^2(2k-1)^2}\right)}_{A_k}\cos[(2k-1)\omega_1 t]$$

$$\begin{cases} A_0=0 & \Rightarrow \text{ Zero mean} \\ B_k=0 & \Rightarrow \text{ is an even function and looks like a cosine function} \end{cases}$$

Take the first six terms and let $A = 10$, $T = 0.1$:

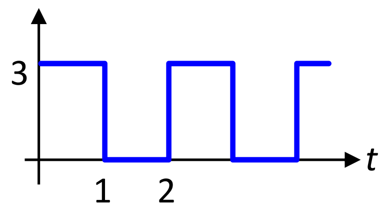
$$\begin{aligned} x(t) &= 8.105\cos(20\pi t) \\ &+ 0.901\cos(3\times 20\pi t) \\ &+ 0.324\cos(5\times 20\pi t) \\ &+ 0.165\cos(7\times 20\pi t) \\ &+ 0.100\cos(9\times 20\pi t) \\ &+ 0.067\cos(11\times 20\pi t) \end{aligned}$$



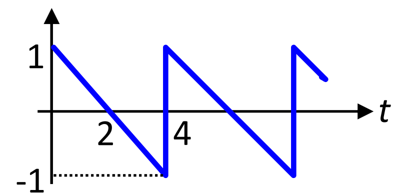


Periodic Signals - Fourier Series

- **Calculating the Fourier Coefficients**:
  - $A_0/2$ represents the average of the signal $x(t)$. It contains the "DC" (zero frequency) component of the signal.

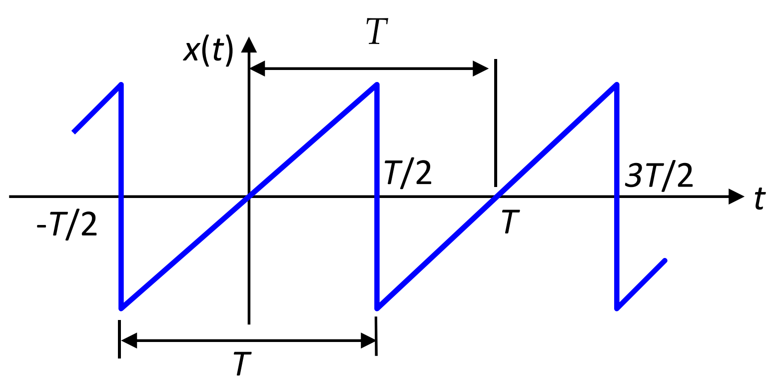


$$\frac{A_0}{2} = \frac{3}{2}$$



$$\frac{A_0}{2} = 0$$

  - When calculating the Fourier coefficients, changing the integral limits will not affect the results, as long as the integration covers one complete period of the signal.



$$A_k = \frac{2}{T}\int_0^T x(t)\cos\left(k\frac{2\pi}{T}t\right)dt$$

$$= \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos\left(k\frac{2\pi}{T}t\right)dt$$

$$= \frac{2}{T}\int_{-T/4}^{3T/4} x(t)\cos\left(k\frac{2\pi}{T}t\right)dt = 0$$

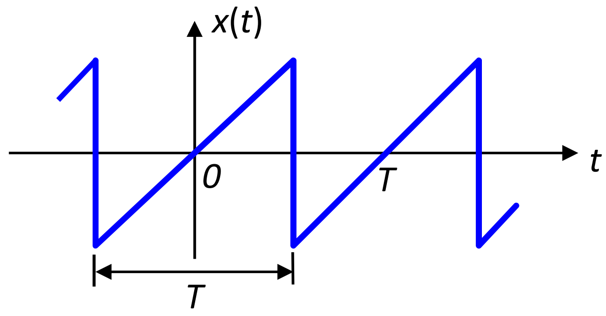
$x(-t) = -x(t) \Rightarrow$ an odd function $\Rightarrow$ looks similar to sine…

Periodic Signals - Fourier Series

- **Calculating the Fourier Coefficients**:
  - Odd functions (signals), for which $x(-t) = -x(t)$, will only contain sine terms, i.e., $A_k = 0$, for $k = 0, 1, 2, \ldots$.

$$\text{If } x(-t) = -x(t) \quad \Rightarrow \quad x(t) = \sum_{k=1}^{\infty} B_k \sin(k\omega_1 t)$$

$$\left(\sin(-t) = -\sin(t) \right)$$

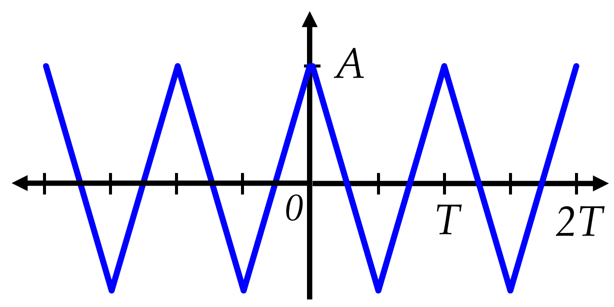


  - Even functions (signals), for which $x(-t) = x(t)$, will only contain cosine terms, i.e. $B_k = 0$, for $k = 1, 2, \ldots$.

$$\text{If } x(-t) = x(t)$$

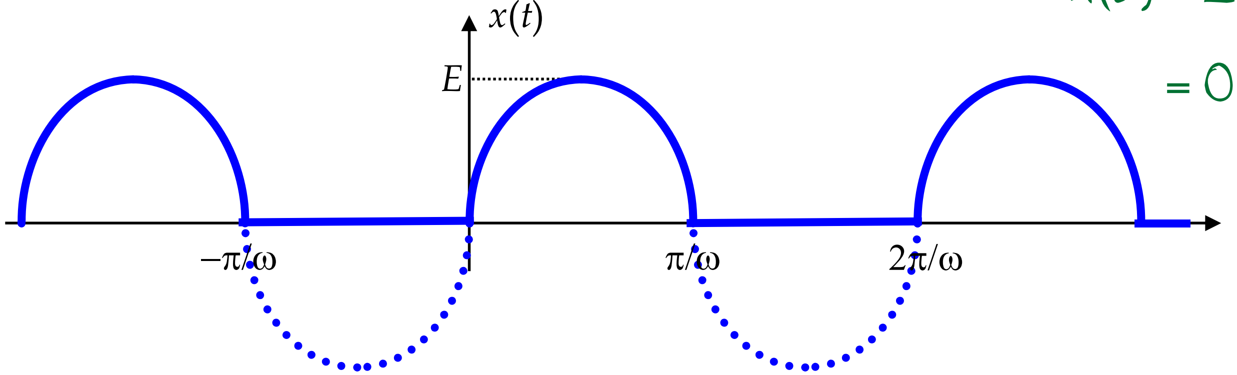
$$\Rightarrow \quad x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\omega_1 t)$$

$$\left(\cos(-t) = \cos(t) \right)$$



Periodic Signals - Fourier Series

Ex: *Half rectified sine wave*



$$x(t) = E\sin(\omega t) \qquad 0 < t < \pi/\omega$$
$$= 0 \qquad -\pi/\omega < t < 0$$

Symbolic:

Q: What is the period, T, of this signal?

*What is the **fundamental frequency** of this signal?*

$$T = \frac{2\pi}{\omega} \Rightarrow f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ or } \omega_1 = \omega$$

Q: Expand the signal into its Fourier series.

→ *Find the corresponding Fourier coefficients*!

Particular numerical example:

Let ω = 1 [rad/s], E = 1 [V]

$$\Rightarrow f = \frac{1}{T} = \frac{\omega}{2\pi} = \frac{1}{2\pi}$$

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Periodic Signals - Fourier Series

Symbolic:

Calculate Fourier Coefficients:

- A_0 :

$$A_0 = \frac{2}{T}\int_0^T x(t)dt = \frac{2\omega}{2\pi}\int_0^{2\pi/\omega} x(t)dt$$

$$= \frac{2\omega}{2\pi}\int_0^{\pi/\omega} E\cdot\sin(\omega t)dt = \frac{\omega}{\pi}\left[-\frac{E}{\omega}\cos(\omega t)\right]_0^{\pi/\omega}$$

$$= \frac{E}{\pi}\left[-\cos(\pi)-(-\cos(0))\right] = \frac{2E}{\pi}$$

- A_k :

$$A_k = \frac{2}{T}\int_0^T x(t)\cdot\cos(k\omega_1 t)dt$$

$$= \frac{2\omega}{2\pi}\int_0^{\pi/\omega} E\cdot\sin(\omega t)\cdot\cos(k\omega_1 t)dt$$

$$k = \underbrace{2m}_{\text{even}} \Rightarrow A_{2m} = \frac{-2E}{\pi(4m^2-1)} \text{ or } A_k = \frac{-2E}{\pi(k-1)(k+1)}$$

$$k = \underbrace{2m-1}_{\text{odd}} \Rightarrow A_{2m-1} = A_k = 0$$

Numeric:

$E = 1, \quad \omega = 1$

$$A_0 = \frac{2}{\pi}$$

For even k :

$$A_k = \frac{-2}{\pi(k-1)(k+1)}$$

Periodic Signals - Fourier Series

Symbolic:

Calculate Fourier Coefficients:

- B_k :

$$B_k = \frac{2}{T}\int_0^T x(t)\cdot\sin(k\omega_1 t)dt$$

$$= \frac{2\omega}{2\pi}\int_0^{\pi/\omega} E\cdot\sin(\omega t)\cdot\sin(k\omega_1 t)dt$$

$$k = 1 \quad \Rightarrow \quad B_1 = \frac{\omega}{\pi}\int_0^{\pi/\omega} E\cdot\sin^2(\omega t)dt \quad = \frac{E}{2}$$

$$k \neq 1 \quad \Rightarrow \quad B_k = 0$$

Numeric:

$E = 1, \quad \omega = 1$

$B_1 = \frac{1}{2}$

$B_2 = B_3 = \cdots = 0$

$A_0 = \frac{2}{\pi} \Rightarrow \frac{A_0}{2} = \frac{1}{\pi}$

Fourier Series Representation:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty}\left[A_k\cos(k\omega_1 t) + B_k\sin(k\omega_1 t)\right] = \frac{A_0}{2} + B_1\sin(\omega t) + \sum_{\text{even } k} A_k\cos(k\omega t)$$

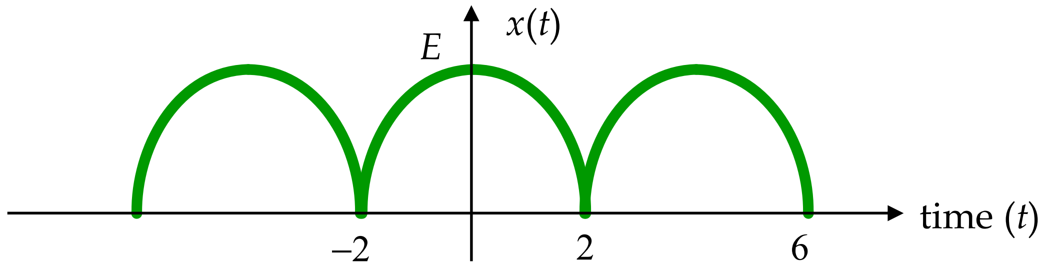
$$= \frac{1}{\pi} + \underbrace{\frac{1}{2}}_{B_1}\sin(\omega t) \quad -\frac{2}{\pi}\left(\frac{1}{3}\cos(2\omega t) + \frac{1}{15}\cos(4\omega t) + \cdots\right)$$

For even k :

$$A_k = \frac{-2}{\pi(k-1)(k+1)}$$

Periodic Signals - Fourier Series

Ex:



$$x(t) = 4 - t^2, \qquad -2 \le t \le 2$$

Q: What is the period, T, of this signal?
What is the fundamental frequency of this signal?

$$T = 4 \qquad \Rightarrow \qquad \omega_1 = \frac{2\pi}{T} = \frac{\pi}{2}$$

Q: Expand the signal into its Fourier series.

$$A_0 = \frac{2}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{16}{3}$$

$B_k = 0 \leftarrow$ WHY?

This is an even function

$$A_k = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(k\omega_1 t)dt = \frac{16}{(k\pi)^2}(-1)^{k+1}$$

$$x(t) = \frac{8}{3} + \frac{16}{\pi^2}\left(\cos\left(\frac{\pi t}{2}\right) - \frac{1}{2^2}\cos\left(\frac{2\pi t}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi t}{2}\right) - \frac{1}{4^2}\cos\left(\frac{4\pi t}{2}\right) + - \cdots\right)$$

Periodic Signals - Fourier Series

- Express a Fourier series using only sine or cosine terms:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty}\left[A_k\cos(k\omega_1 t) + B_k\sin(k\omega_1 t)\right]$$

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\Downarrow$$

$$M\cos(\omega t + \varphi)$$

$$= M\cos(\omega t)\cos(\varphi) - M\sin(\omega t)\sin(\varphi)$$

$$= \underbrace{M\cos(\varphi)}_{A_k}\cdot\cos(\omega t) \underbrace{-M\sin(\varphi)}_{B_k}\cdot\sin(\omega t)$$

$$= \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k\cos(k\omega_1 t + \varphi_k) \quad \leftarrow \text{cosine series}$$

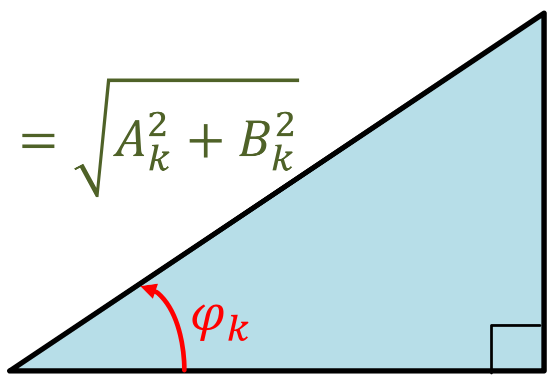
$$\cos(\alpha) = \sin(\alpha + 90^\circ), \forall\alpha$$

$$= \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k\sin(k\omega_1 t + \beta_k) \quad \leftarrow \text{sine series}$$

$$\beta_k = \varphi_k + 90^\circ$$

$$A_k - jB_k = M_k e^{j\varphi_k}$$

$$M_k = \sqrt{A_k^2 + B_k^2}$$



$$-B_k = M_k\sin(\varphi_k)$$

$$\varphi_k = \angle(A_k - jB_k)$$
$$= \text{atan2}(-B_k, A_k)$$

$$A_k = M_k\cos(\varphi_k)$$

Periodic Signals - Fourier Series

- Express a Fourier series in Complex Form:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} [A_k \cos(k\omega_1 t) + B_k \sin(k\omega_1 t)]$$

$$= \sum_{k=-\infty}^{\infty} C_k \cdot e^{jk\omega_1 t}$$

Recall that $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$

$$\Rightarrow \begin{cases} \cos(\omega t) = \frac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) = \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{cases}$$

where

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cdot e^{-jk\omega_1 t}\, dt$$

or

$$C_k = \tfrac{1}{2}(A_k - jB_k) = \tfrac{1}{2} M_k e^{j\varphi_k}$$

$$|C_k| = \tfrac{1}{2}\sqrt{A_k^2 + B_k^2} = \tfrac{1}{2} M_k$$

$$\text{Arg}[C_k] = \varphi_k$$

$$C_{-k} = \frac{1}{2}(A_k + jB_k) = \overline{C_k}$$

PURDUE MECHANICAL ENGINEERING

Periodic Signals - Fourier Series

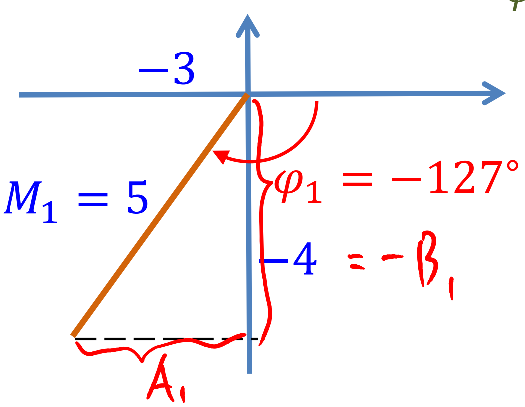
Ex: write the following periodic signal in a sine and cosine series form and plot its magnitude and phase vs. frequency

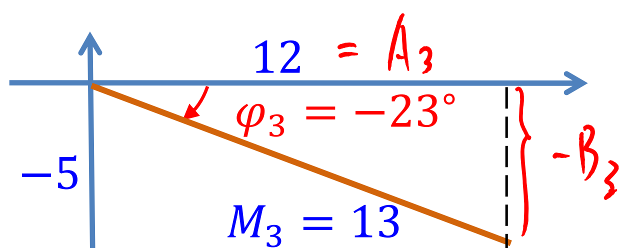
$$x(t) = \underbrace{-3}_{A_1}\cos(24\pi t) + \underbrace{4}_{B_1}\sin(24\pi t) + \underbrace{12}_{A_3}\cos(72\pi t) + \underbrace{5}_{B_3}\sin(72\pi t)$$

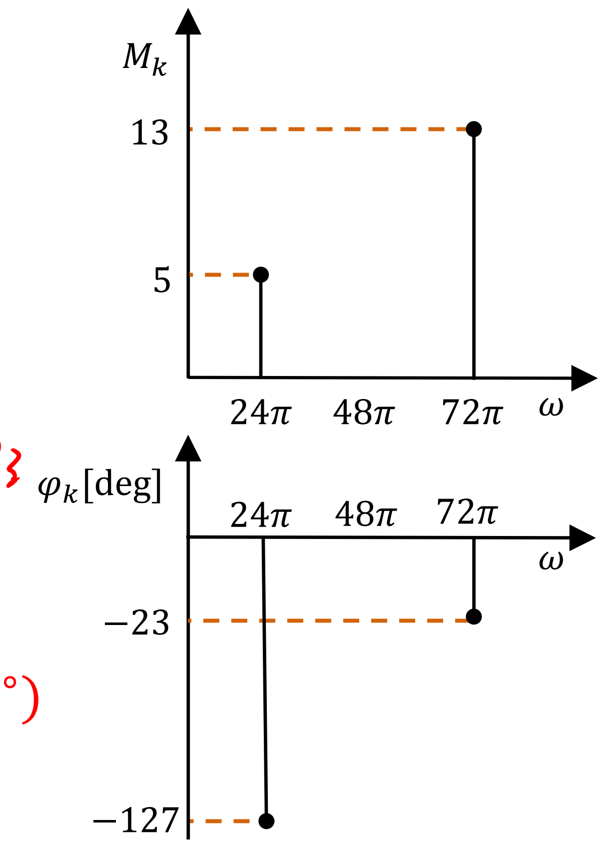
$$\omega_1 = 24\pi\left[\frac{\text{rad}}{\text{sec}}\right] \qquad \omega_3 = 3\omega_1 = 72\pi\left[\frac{\text{rad}}{\text{sec}}\right]$$

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k\cos(k\omega_1 t + \varphi_k) \quad \leftarrow \text{cosine series}$$

$$\varphi_k = \angle(A_k - jB_k) = \text{atan2}(-B_k, A_k)$$







$$\Longrightarrow \quad x(t) = 5\cos(24\pi t - 127°) + 13\cos(72\pi t - 23°)$$

$$= 5\sin(24\pi t - 37°) + 13\sin(72\pi t + 67°)$$

PURDUE MECHANICAL ENGINEERING

Periodic Signals - Fourier Series

Ex: write the previous periodic signal in a complex Fourier series form and plot its magnitude and phase vs. frequency

$$x(t) = -3\cos(24\pi t) + 4\sin(24\pi t) + 12\cos(72\pi t) + 5\sin(72\pi t)$$

$\omega_1 = 24\pi$ $\omega_3 = 3\omega_1 = 72\pi$

$$x(t) = \sum_{k=-\infty}^{\infty} C_k \cdot e^{jk\omega_1 t}$$

$$C_k = \tfrac{1}{2}(A_k - jB_k) = \tfrac{1}{2}M_k e^{j\varphi_k}$$

$$C_1 = \tfrac{1}{2}(A_1 - jB_1) = \tfrac{1}{2}(-3 - j4) = \tfrac{5}{2}e^{-j127^\circ}$$

$$\Rightarrow \quad C_{-1} = \overline{C}_1 = \tfrac{1}{2}(-3 + j4) = \tfrac{5}{2}e^{j127^\circ}$$

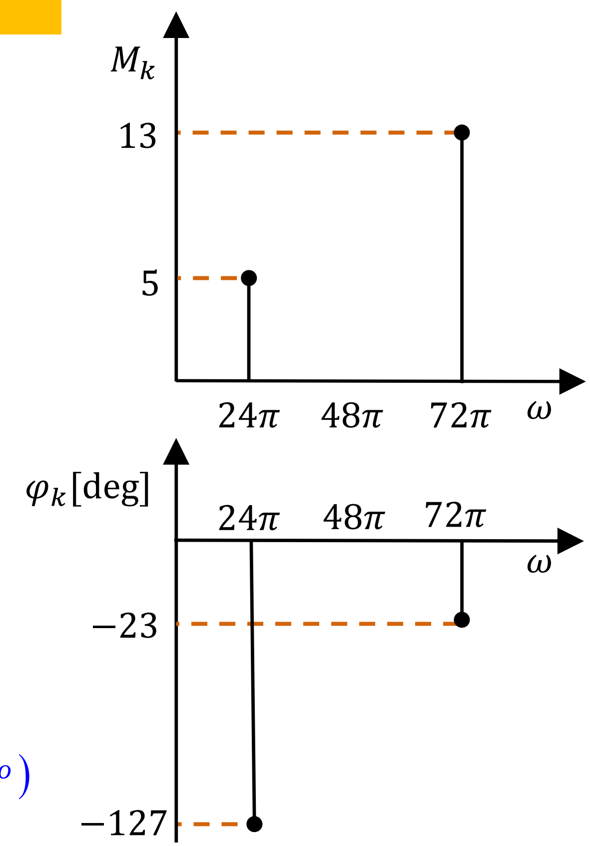
$$C_3 = \tfrac{1}{2}(A_3 - jB_3) = \tfrac{1}{2}(12 - j5) = \tfrac{13}{2}e^{-j23^\circ}$$

$$\Rightarrow \quad C_{-3} = \overline{C}_3 = \tfrac{1}{2}(12 + j5) = \tfrac{13}{2}e^{j23^\circ}$$

$$\Longrightarrow$$

$$x(t) = C_{-3}e^{-j3\omega_1 t} + C_{-1}e^{-j\omega_1 t} + C_1 e^{j\omega_1 t} + C_3 e^{j3\omega_1 t}$$

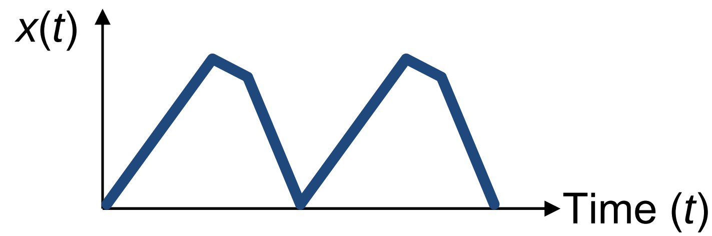
$$= \frac{13}{2}e^{-j\left(72\pi t - 23^o\right)} + \frac{5}{2}e^{-j\left(24\pi t - 127^o\right)} + \frac{5}{2}e^{j\left(24\pi t - 127^o\right)} + \frac{13}{2}e^{j\left(72\pi t - 23^o\right)}$$



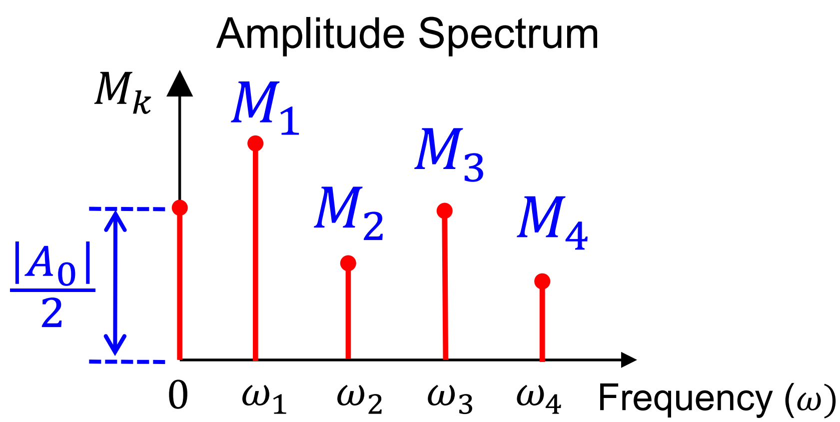
Periodic Signals - Frequency Spectra

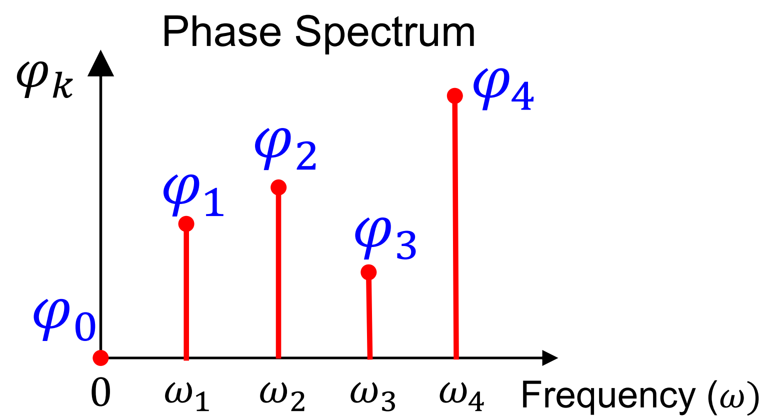
- Time Domain:

$$x(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k \cos\left(k\omega_1 t + \varphi_k\right)$$



- Frequency Domain:
  - ➢ Amplitude and Phase of Fourier coefficients vs. Frequency





Frequency Spectra

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Periodic Signals - Frequency Spectra

Ex: Plot the amplitude and phase spectra of the following signal

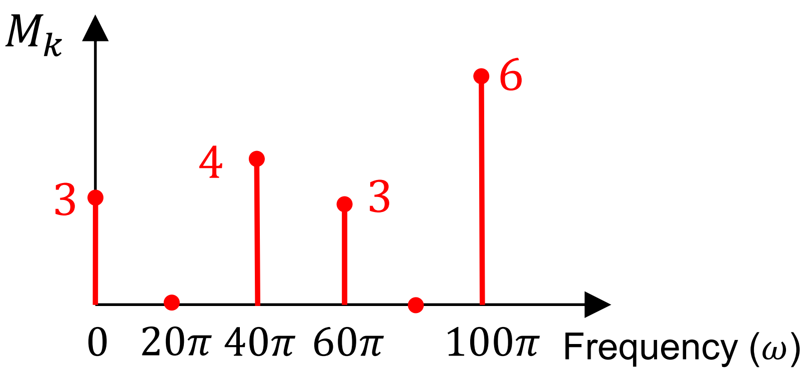
$$x(t) = -3 + 4\cos\left(40\pi t + \frac{\pi}{3}\right) + 3\cos\left(60\pi t - \frac{\pi}{2}\right) + 6\,\underline{\sin}\left(100\pi t - \frac{\pi}{4}\right)$$

$$\| \quad \cos\left(100\pi t - \frac{\pi}{4} - \frac{\pi}{2}\right) = \cos\left(100\pi t - \frac{3\pi}{4}\right)$$

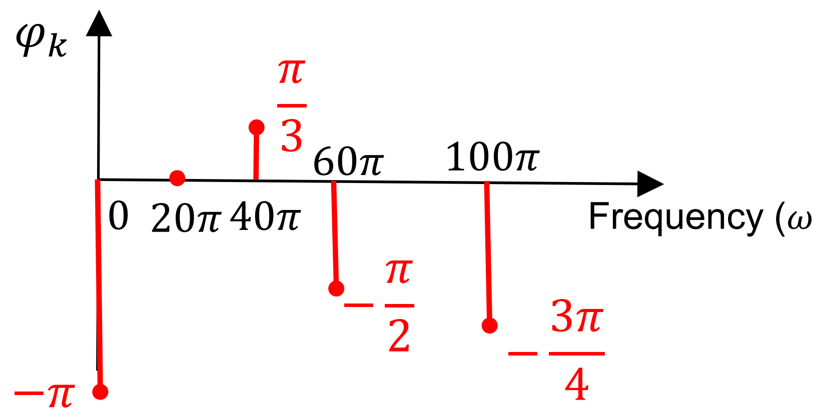
Fundamental frequency is greatest common divisor of the signal's frequency components:

$$\omega_1 = 20\pi \left[\frac{\text{rad}}{\text{sec}}\right] \quad \Longrightarrow f_1 = \frac{\omega_1}{2\pi} = 10\ [\text{Hz}]$$

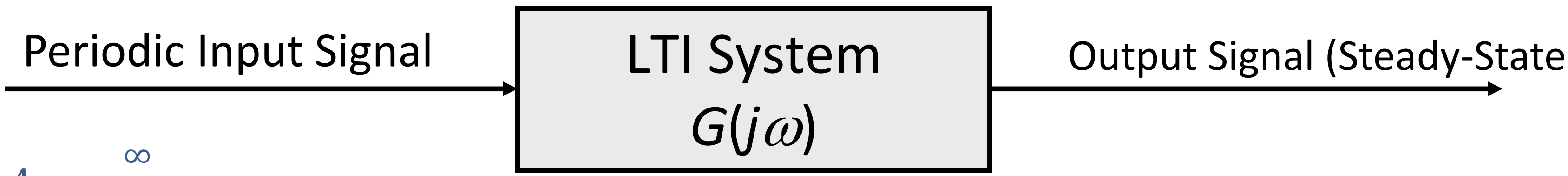
Amplitude Spectrum



Phase Spectrum



Signals Through Systems



$$u(t) = \frac{A_0}{2} + \sum_{k=1}^{\infty} M_k \cos(k\omega_1 t + \varphi_k)$$

$$y_{ss}(t) = ?$$

Completely characterized by

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$$

$\Rightarrow$

$$y_{ss}(t) = \frac{A_0}{2} G(j0) + \sum_{k=1}^{\infty} M_k |G(j\omega_k)| \cos(\underbrace{k\omega_1}_{\omega_k} t + \varphi_k + \angle G(j\omega_k))$$

$$= \frac{A_{y0}}{2} + \sum_{k=1}^{\infty} M_{yk} \cos(\underbrace{k\omega_1}_{\omega_k} t + \varphi_{yk})$$

$$A_{y0} = A_0 G(j0)$$

$$M_{yk} = M_k |G(j\omega_k)|$$

$$\varphi_{yk} = \varphi_k + \angle G(j\omega_k)$$

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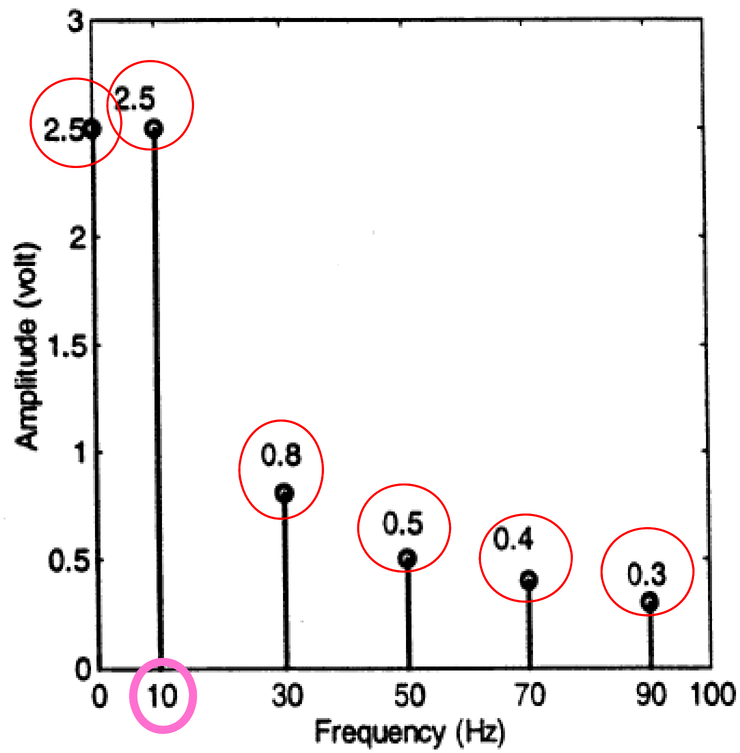
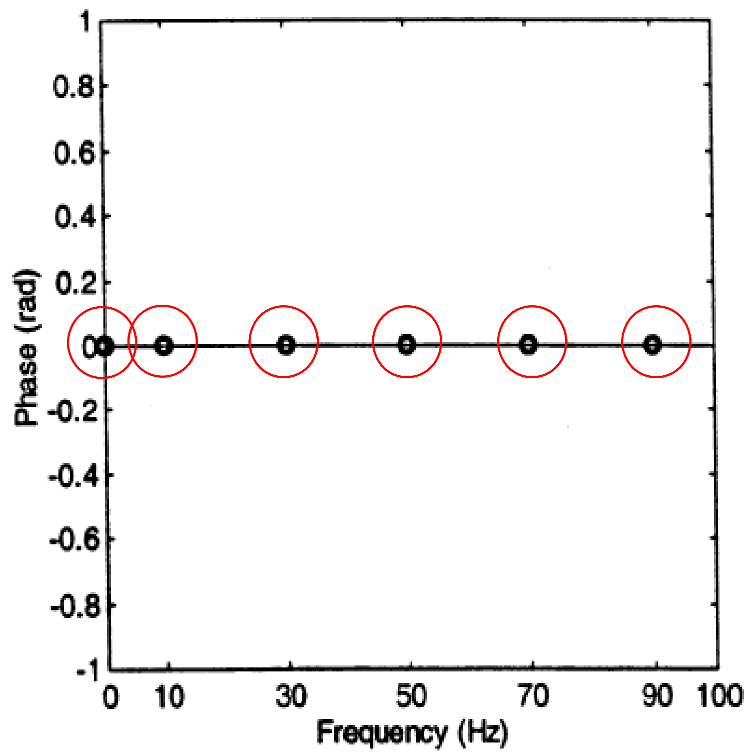
Signals Through Systems

Ex: An inkjet nozzle "firing" signal has the following frequency spectra, write down its time-domain expression



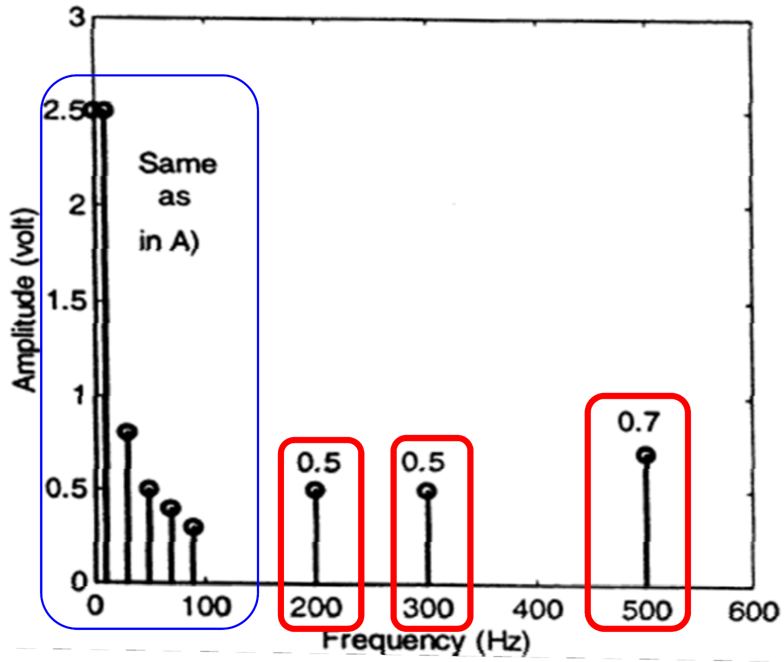


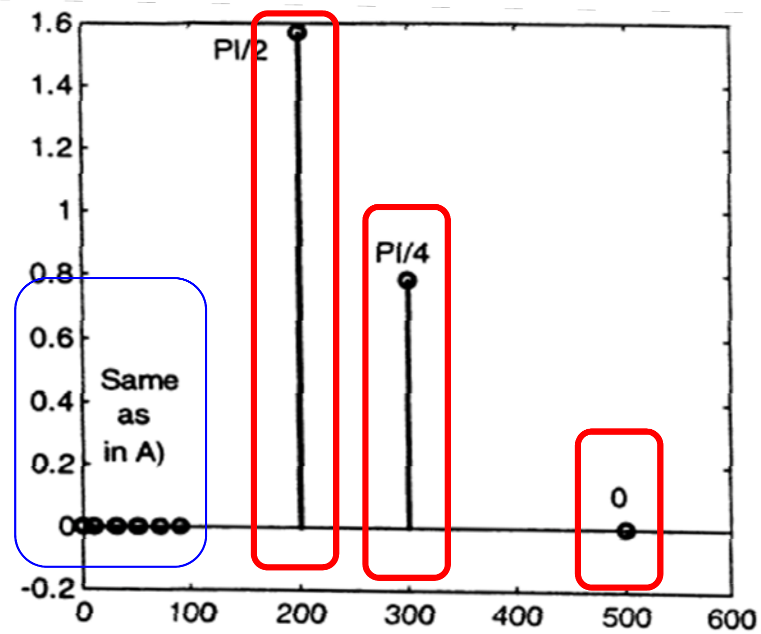
$$u(t) = 2.5 + 2.5\times\cos(\underbrace{20\pi t}_{10\text{ Hz}}) + 0.8\times\cos(60\pi t) + 0.5\times\cos(100\pi t)$$

$$+ 0.4\cos(140\pi t) + 0.3\cos(180\pi t)$$

Signals Through Systems

Ex: After going through the PC board on the pen carriage, the spectrum plots of the signal look like:





Write down the time-domain expression of the augmented signal:

$$u_a(t) = 2.5 + 2.5\cos(20\pi t) + 0.8\cos(60\pi t) + 0.5\cos(100\pi t) + 0.4\cos(140\pi t) + 0.3\cos(180\pi t)$$

$$+ 0.5\cos\left(400\pi t + \frac{\pi}{2}\right) + 0.5\cos\left(600\pi t + \frac{\pi}{4}\right) + 0.7\cos(1000\pi t)$$

PURDUE MECHANICAL ENGINEERING

Signals Through Systems

Ex: If the augmented signal is to pass through a filter with the following frequency response function:

$$G(j\omega) = \frac{1}{0.00318(j\omega) + 1}$$

$$|G(j\omega)| = \frac{1}{\sqrt{(0.00318\omega)^2 + 1}}$$

$$|G(j20\pi)| = 0.9806$$

$$\angle G(j\omega) = -\tan^{-1}(0.00318\omega)$$

$$\angle G(j20\pi) = -0.1972 \text{ [rad]}$$

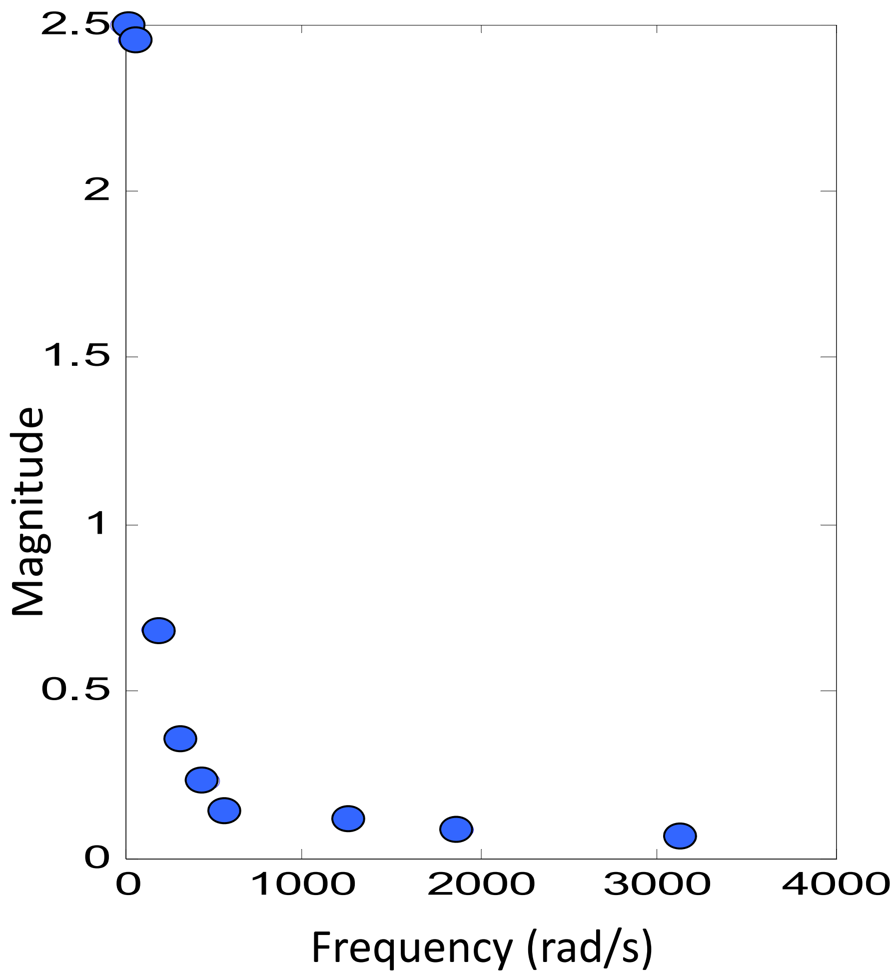
Obtain the steady-state output signal:

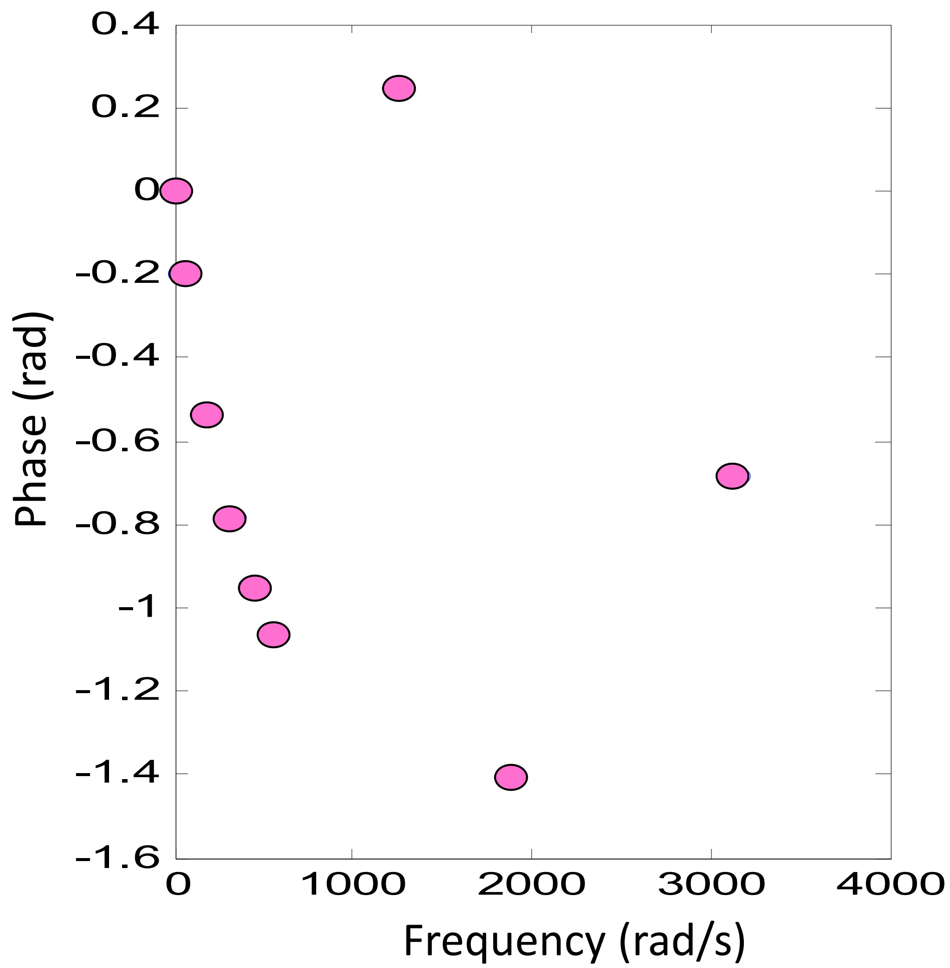
$$u_a(t) = 1.25 + 2.5\cos(20\pi t) + 0.8\cos(60\pi t) + 0.5\cos(100\pi t) + 0.4\cos(140\pi t) + 0.3\cos(180\pi t) + 0.5\cos\left(400\pi t + \frac{\pi}{2}\right) + 0.5\cos\left(600\pi t + \frac{\pi}{4}\right) + 0.7\cos(1000\pi t)$$

$$y(t) = 2.5 + \underbrace{2.4515}_{2.5\,|G(j20\pi)|}\cos(20\pi t \underbrace{- 0.1972}_{\angle G(j20\pi)}) + 0.6862\cos(60\pi t - 0.54) + 0.3537\cos(100\pi t - 0.7849) + 0.2326\cos(140\pi t - 0.9501) + 0.1458\cos(180\pi t - 1.0633) + 0.1214\cos(400\pi t + 0.2452) + 0.0823\cos(600\pi t - 0.6201) + 0.0697\cos(1000\pi t - 1.471)$$

Signals Through Systems

Ex: Plot the frequency spectra of the output signal

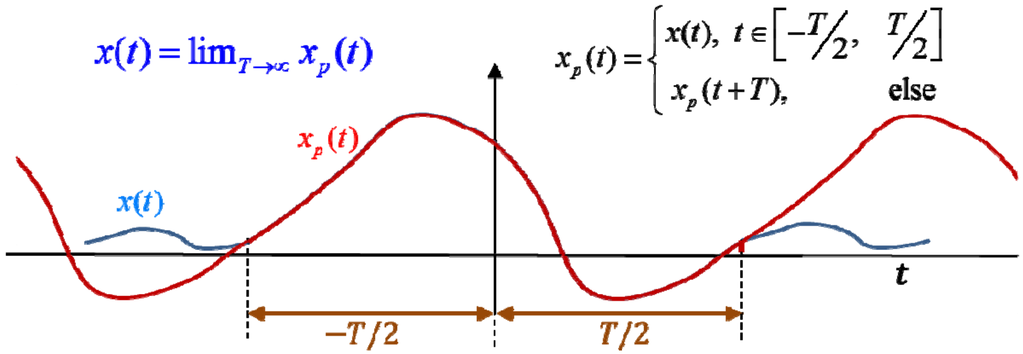




PURDUE MECHANICAL ENGINEERING

Non-Periodic Signals - Fourier Transform

Loosely speaking, a non-periodic signal $x(t)$ can be studied as the limit of a periodic signal $x_p(t)$ of period T having the same value as $x(t)$ during [-T/2, T/2] when $T \to \infty$:



Fourier Series of $x_p(t)$:

$$x_p(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_1 t} \qquad C_k = \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-jk\omega_1 t} dt = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j\omega t} dt$$

When $T \to \infty$:

$$\omega = k\omega_1 \quad \to \quad \pm\infty \quad \text{as} \quad k \to \pm\infty$$

$$\Delta\omega = \omega_1 = \frac{2\pi}{T} \quad \to \quad 0$$

Thus

$$x(t) = \lim_{T\to\infty} x_p(t) = \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_1 t} = \frac{1}{2\pi} \lim_{T\to\infty} \sum_{k=-\infty}^{\infty} \underbrace{\frac{2\pi C_k}{\omega_1}}_{X_T(j\omega)} e^{j\omega t} \Delta\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega, \qquad X(j\omega) = \lim_{T\to\infty} X_T(j\omega)$$

PURDUE MECHANICAL ENGINEERING

Non-Periodic Signals - Fourier Transform

Note

$$X(j\omega) = \lim_{T\to\infty} X_T(j\omega) = \lim_{T\to\infty} \frac{2\pi}{\omega_1}\cdot\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt = \lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$\Rightarrow$

Fourier Transform:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Inverse Fourier Transform:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

Time-domain Signal

$$\{x(t),\ t\in(-\infty,\infty)\}$$

$$\xrightarrow{\mathcal{F}\{\cdot\}} \qquad \xleftarrow{\mathcal{F}^{-1}\{\cdot\}}$$

Frequency-domain Signal

$$\{X(j\omega),\ \omega\in(-\infty,\infty)\}$$

Signals Through Systems

Input Signal → **LTI System** $G(j\omega)$ → Output Signal (Steady-State)

$$\{u(t),\ t \in (-\infty, \infty)\}$$

$$\{U(j\omega),\ \omega \in (-\infty, \infty)\}$$

$$y_{ss}(t) = ?$$

Completely characterized by

$$G(j\omega) = |G(j\omega)| e^{j\angle G(j\omega)}$$

$\Rightarrow$

$$Y_{ss}(j\omega) = G(j\omega)U(j\omega)$$

$$y_{ss}(t) = \mathcal{F}^{-1}\{Y_{ss}(j\omega)\}$$

$$|Y_{ss}(j\omega)| = |G(j\omega)||U(j\omega)|$$

$$\angle Y_{ss}(j\omega) = \angle G(j\omega) + \angle U(j\omega)$$

PURDUE MECHANICAL ENGINEERING