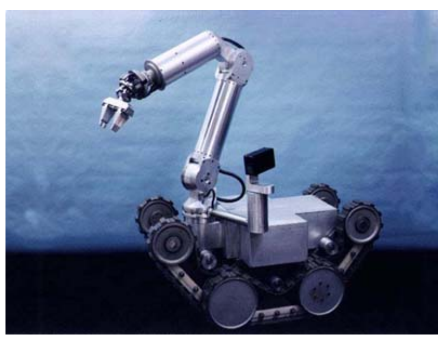
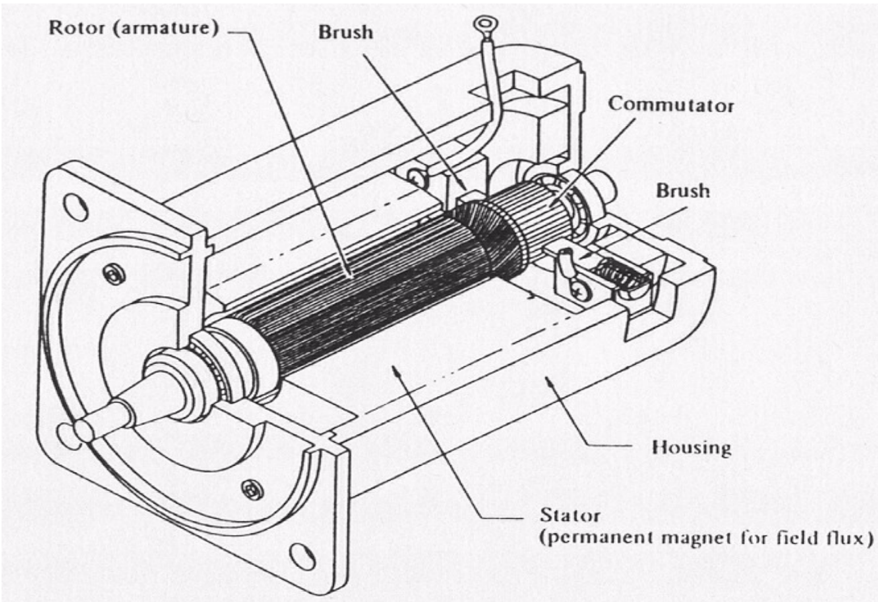
Electro-Mechanical Systems

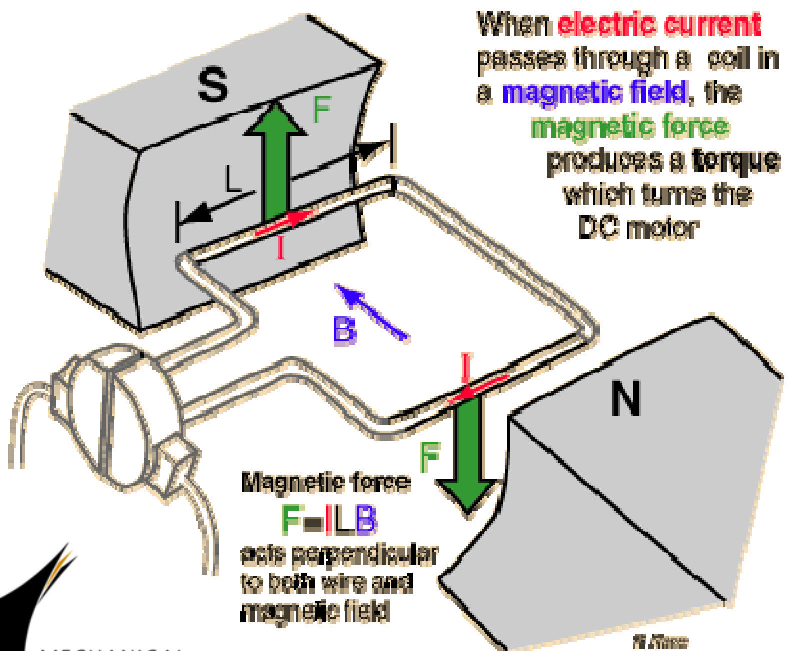
- **DC Motors**
  - Principles of Operation
  - Modeling (EOM)
- **Block Diagram Representations**
  - A Convenient Graphic Representations of Interconnections among various Subsystems of a complex system described by Equations in *s* - Domain
  - Block Diagram Representation of DC Motors
- **Example**



PURDUE MECHANICAL ENGINEERING

DC Motors





Motors are actuation devices (actuators) generating ***torque*** as actuation.

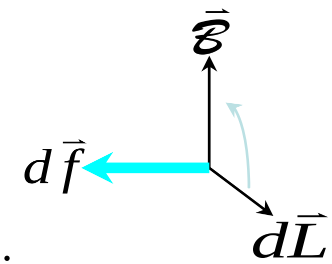
- **Terminology**
  - <u>*Rotor*</u> : the rotating part of the motor.
  - <u>*Stator*</u> : the stationary part of the motor.
  - <u>*Field System*</u> : the part of the motor that provides the magnetic flux.
  - <u>*Armature*</u> : the part of the motor which carries current that interacts with the magnetic flux to produce torque.
  - <u>*Brushes*</u> : the part of the electrical circuit through which the current is supplied to the armature.
  - <u>*Commutator*</u> : the part of the rotor that is in contact with the brushes.

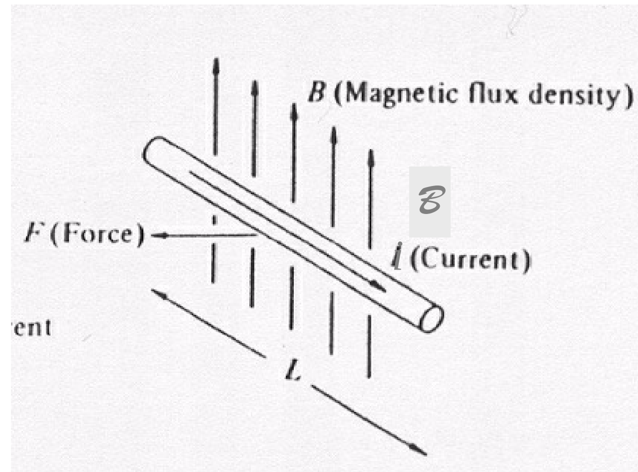
DC Motors - Principles of Operation

• Torque Generation

Needs three elements:

- Magnetic field
- Conductor
- Current





Force will act on a conductor in a magnetic field with current flowing through the conductor:

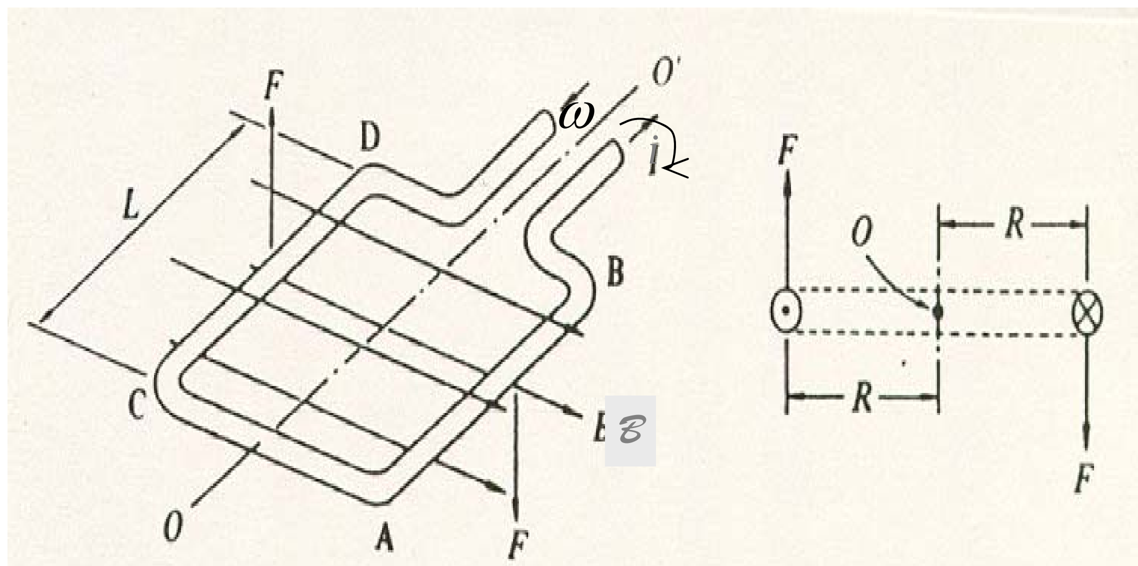
$$d\vec{f} = i_a \cdot d\vec{L} \times \vec{\mathcal{B}}$$

Integrate over the entire length:

$$f = i_a L B$$

Total torque generated:

$$\tau_{Coil} = 2f \cdot R$$
$$= 2 i_a L B R$$
$$= 2 B L R \cdot i_a$$



(a) A signal coil

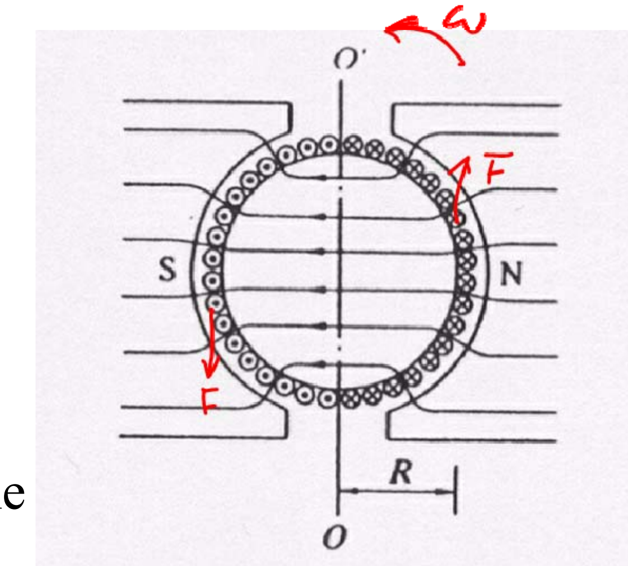
(b) Around axis

DC Motors - Principles of Operation

Let N be the number of coils in the motor. The total torque generated from the N coils is:

$$\tau_m = N \cdot (2 \cdot \underbrace{i_a \cdot \mathcal{B} \cdot L}_{F} \cdot R)$$

$$= \underbrace{N \cdot (2 \cdot \mathcal{B} \cdot L \cdot R)}_{K_T} \cdot i_a$$



For a given motor, (N, $\mathcal{B}$, L, R) are fixed. We can define

$$K_T = 2 \cdot N \cdot \mathcal{B} \cdot L \cdot R \qquad [\mathrm{Nm} / \mathrm{A}]$$

as the *Torque Constant* of the motor.

The torque generated by a DC motor is proportional to the armature current i_a :

$$\tau_m = K_T \cdot i_a$$

For a DC motor, it is desirable to have a large K_T. However, size and other physical limitations often limits the achievable

Large K_T :

– *Large (N, L, R).*

(N, L, R) is limited by the size and weight of the motor.

– *Large $\mathcal{B}$:*

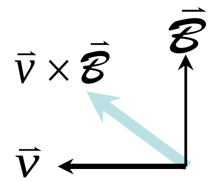
Need to understand the methods of generating flux ...

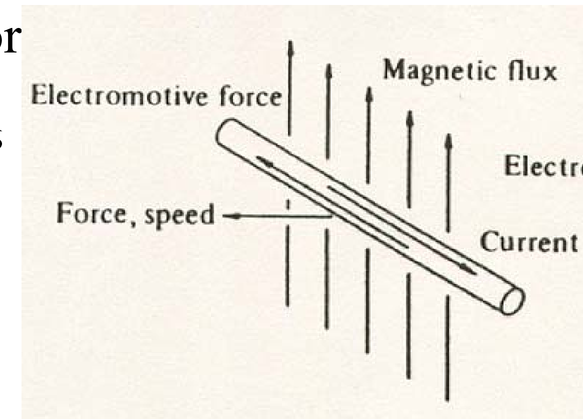
DC Motors - Principles of Operation

- Back-EMF Generation

Electromotive force (EMF) is generated in a conductor moving in a magnetic field:

$$de_{emf} = (\vec{v} \times \vec{\mathcal{B}}) \cdot d\vec{L}$$





Integrate over the entire length L:

$$e_{emf} = vBL$$

Since the N armature coils are in series, the total EMF is:

$$E_{emf} = 2N\underbrace{(R\omega)}_{v}\mathcal{B}L = \underbrace{2NR\mathcal{B}L}_{K_b}\omega$$

Define the *Back-EMF Constant* K_b : $K_b = 2 \cdot N \cdot R \cdot \mathcal{B} \cdot L$ $\quad$ [V / (rad / sec)]

The Back-EMF due to the rotation of the motor armature is opposing the applied voltage and is proportional to the angular speed ω of the motor:

$$E_{emf} = K_b \cdot \omega$$

Note: ***$K_T = K_b$ is true only if SI unit is used !***

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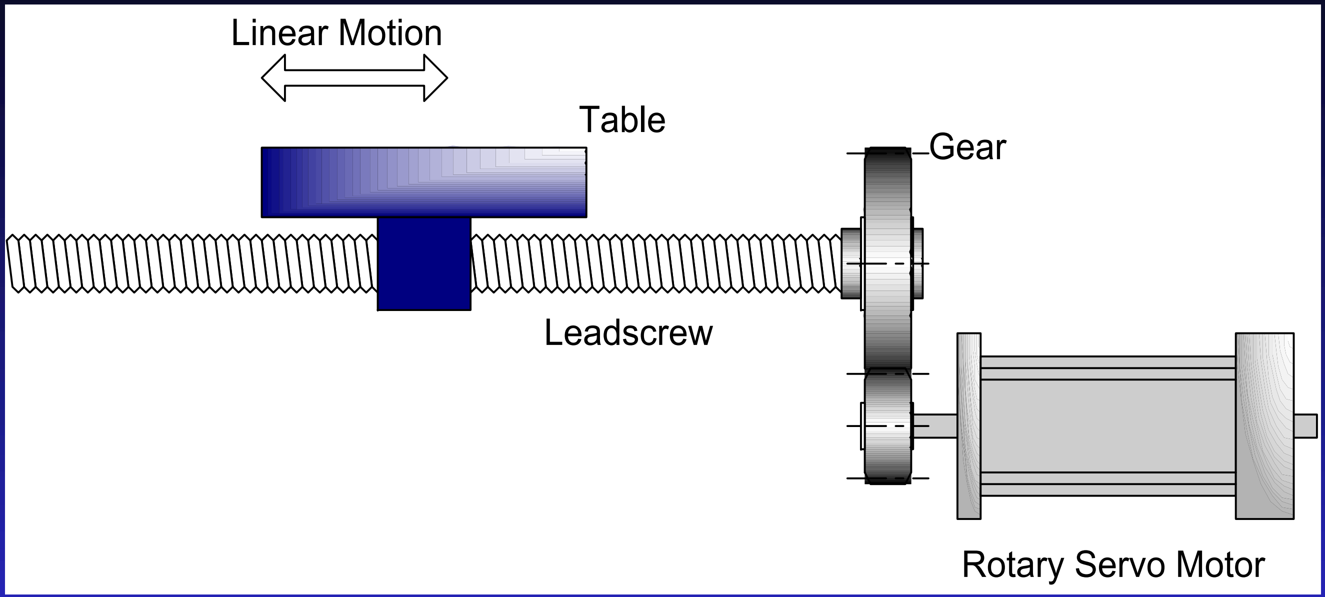
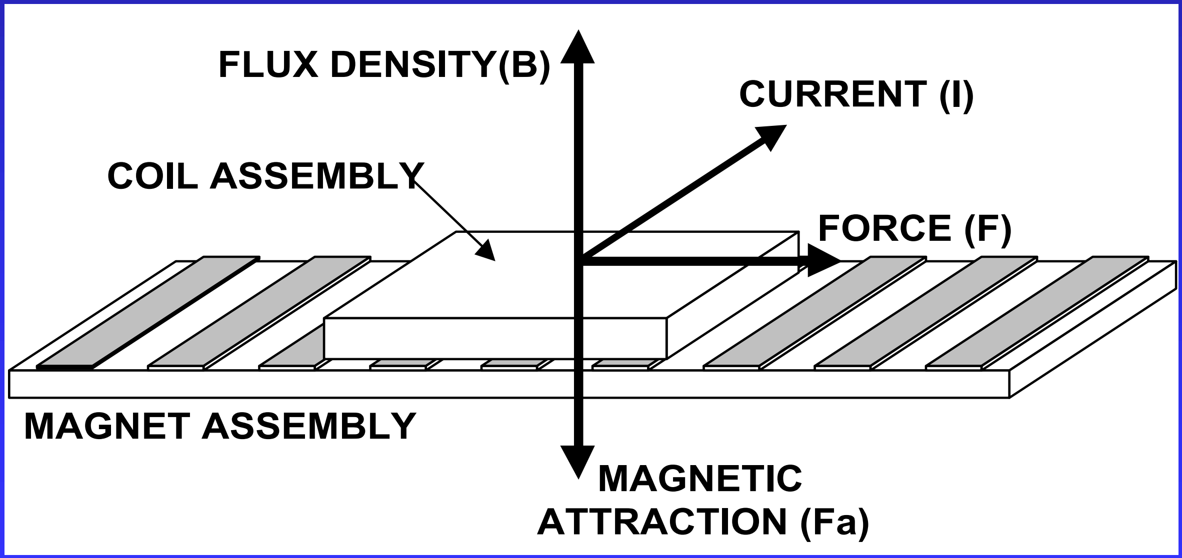




Linear Motor vs. Rotary Motor

Advantages of Linear Motor Drive Systems

- Mechanical simplicity (no mechanical transmission mechanisms), higher reliability, and longer lifetime
- No backlash and less friction, resulting in the potential of having high load positioning accuracy
- No mechanical limitations on achievable acceleration and velocity
- Bandwidth is only limited by encoder resolution, measurement noise, calculation time, and frame stiffness

Difficulties in Control of Linear Motors

- Model Uncertainties

  Parametric uncertainties (e.g., *load inertia*)

  Discontinuous disturbances (e.g., *Coulomb friction*); external disturbances (e.g., *cutting force*)

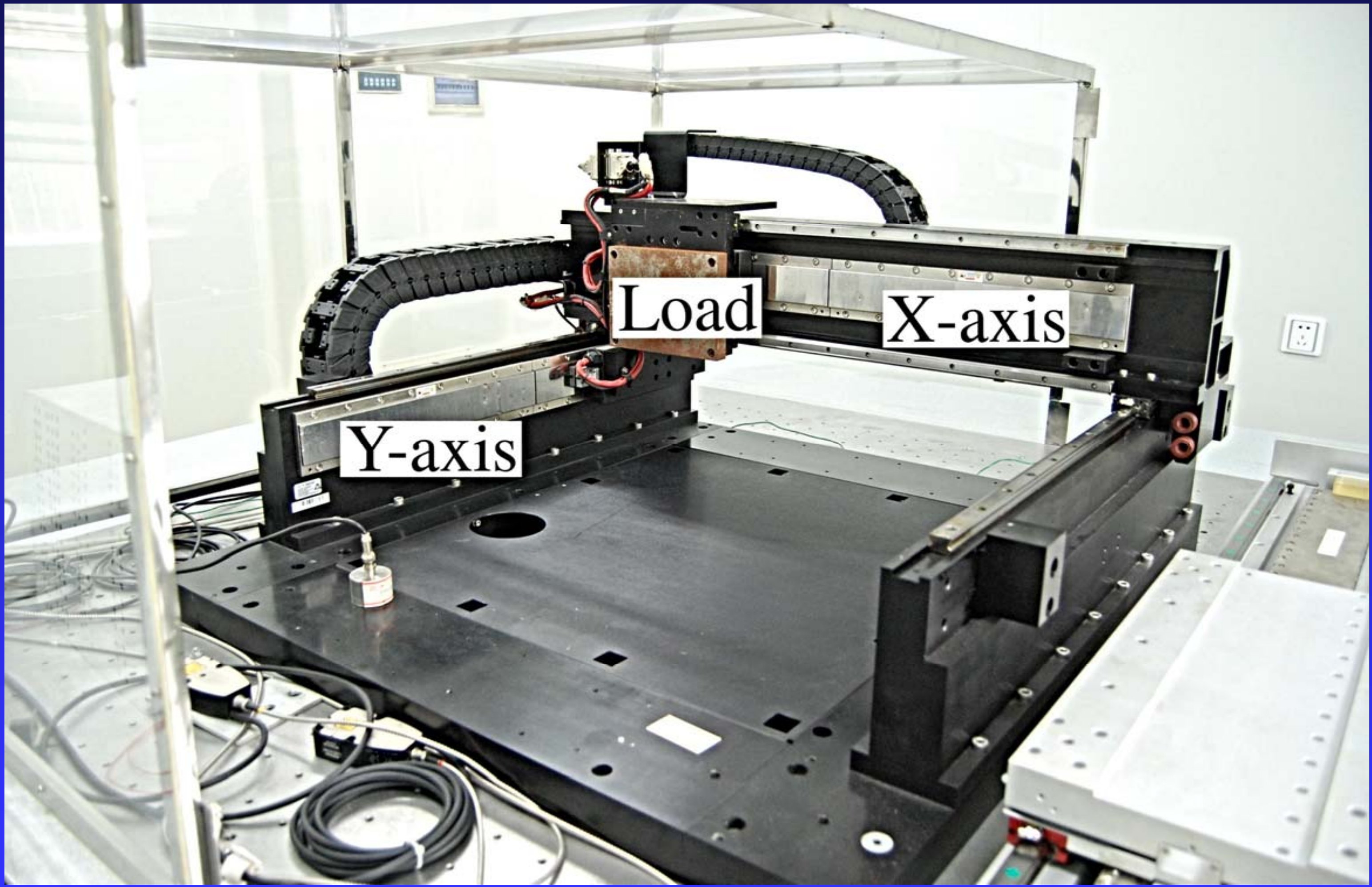
- Drawback of without mechanical transmissions

  Gear reduction reduces the effect of model uncertainties and external disturbance

- Significant uncertain nonlinearities due to position dependent electro-magnetic *force ripples* (e.g. iron-core linear motors)

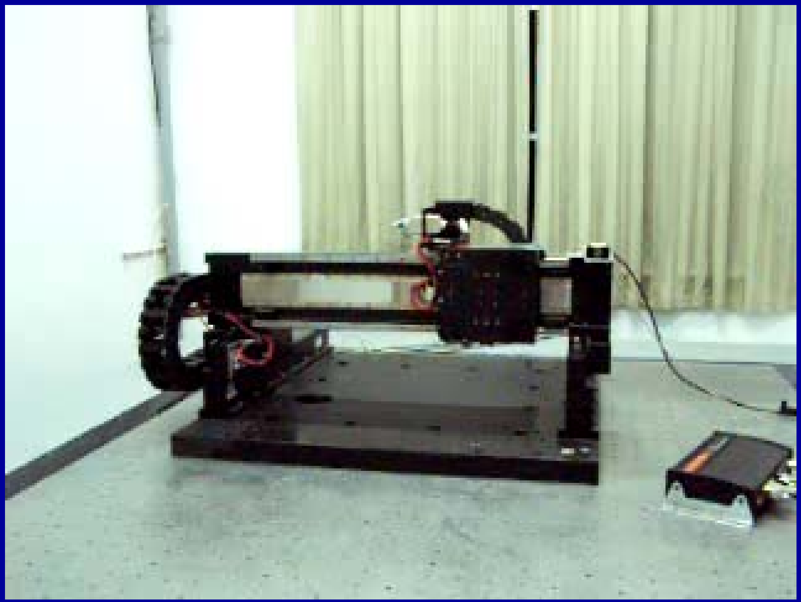
- Implementation issues (e.g., *measurement noise*)

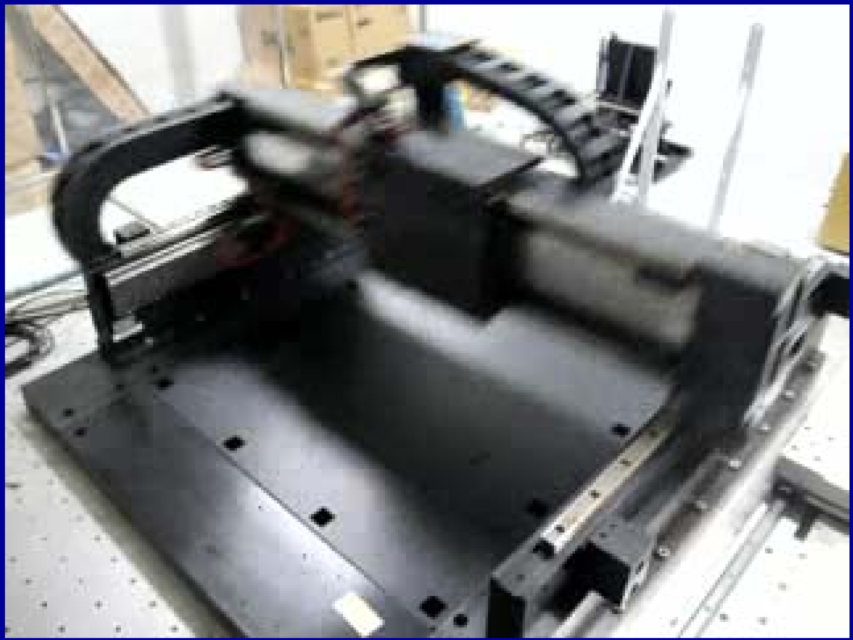
A Biaxial Industrial Gantry System



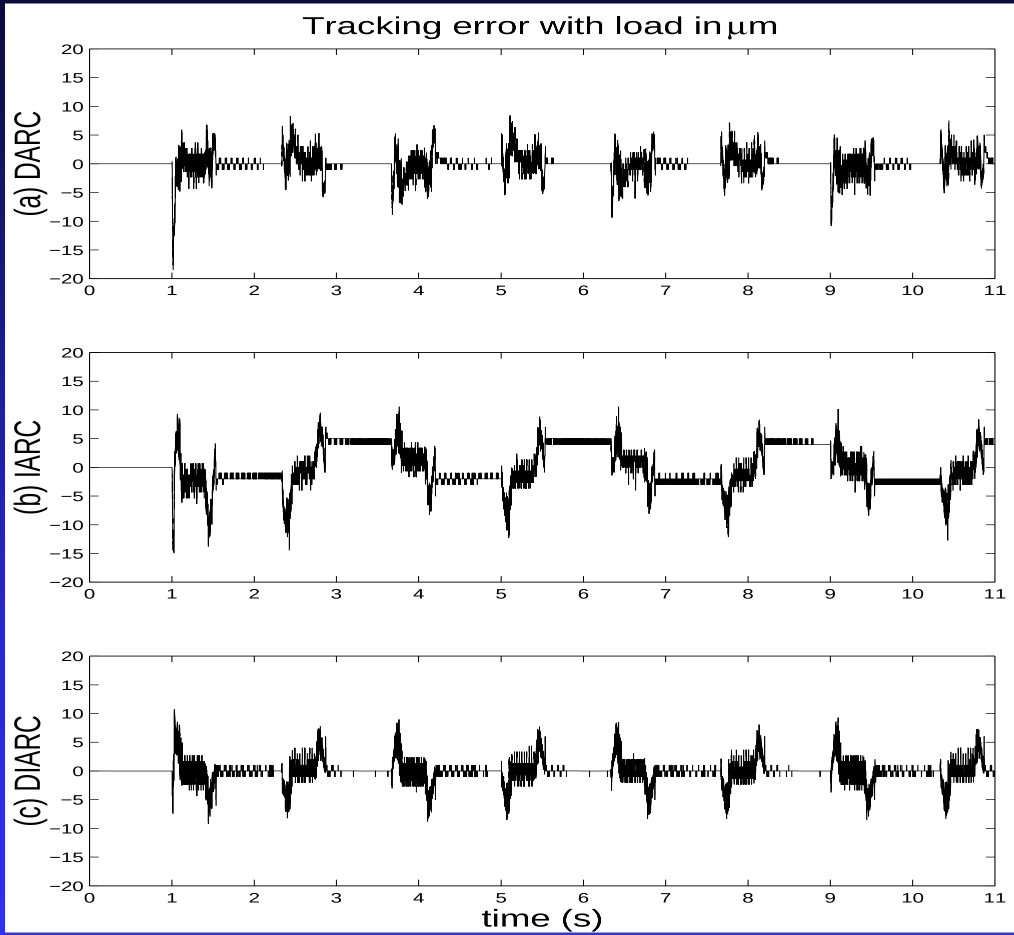
Linear encoder resolution: $0.5\mu m$ **Laser interferometer resolution:** $20nm$

High Speed/Acceleration Experiments





Tracking Errors for Point-to-Point Trajectory $(a_{max} = 12\,m/\sec^2, v_{max} = 1\,m/\sec)$

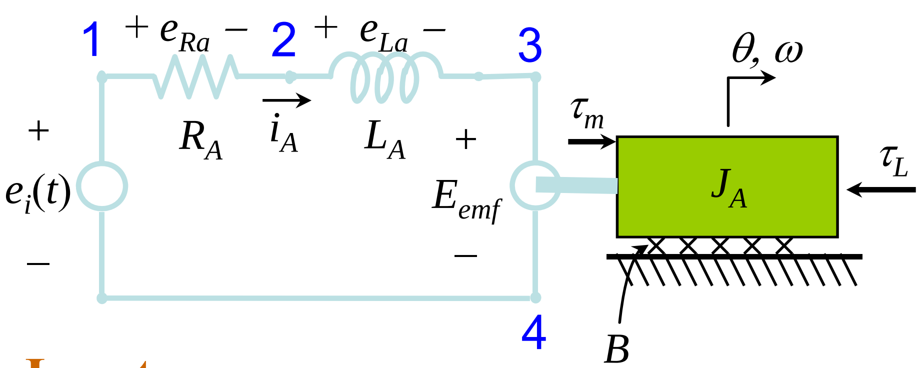


The above results demonstrate the excellent robust tracking performance of the proposed ARC algorithms – Tracking errors are mostly within 10 μm with final tracking error around the encoder resolution of 1 μm for both loaded and unloaded cases

DC Motors - Modeling

$i_A R_A + L\frac{di_A}{dt} + e_{emf} = e_i$

Schematic



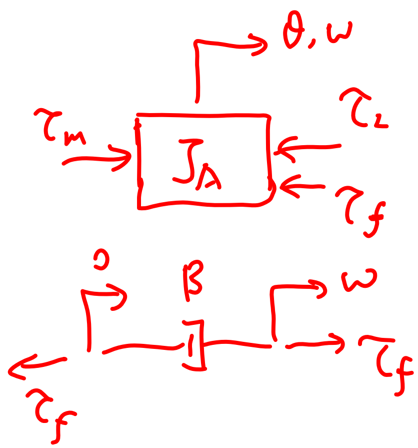
Inputs:

$e_i(t)$, τ_L

Outputs:

θ

FBD:



Element Equations:

Electrical Subsystem

$e_i = e_{12} + e_{23} + e_{34}$

$e_{12} = R_A i_A$ $\quad e_{34} = E_{emf}$

$e_{23} = L\frac{di_A}{dt}$

Mechanical Subsystem:

$J_A\ddot{\theta} = \tau_m - \tau_L - \tau_f$

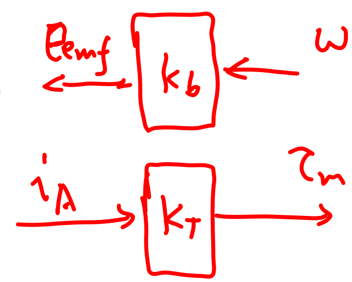
$\tau_f = B\omega$

$\Rightarrow J_A\ddot{\theta} + B\dot{\theta} = \tau_m - \tau_L$

Interconnection Laws:

$\tau_m = K_T i_A$

$E_{emf} = K_b \omega$



Unknowns:

DC Motors - Modeling

$I_A(s) = \mathcal{L}\{i_A\}$. $E_i(s) = \mathcal{L}\{e_i(t)\}$

$E_{emf}(s) = \mathcal{L}[e_{emf}]$, $T_L(s) = \mathcal{L}[\tau_L(t)]$

$T_m(s) = \mathcal{L}[\tau_m]$

Derive I/O Model:

$$\begin{cases} R_A I_A(s) + L_A s I_A(s) + E_{emf}(s) = E_i(s) \\ J_A s^2 \theta(s) + B s \theta(s) = T_m(s) - T_L(s) \\ T_m(s) = K_T I_A(s) \\ E_{emf}(s) = K_b s \theta(s) \end{cases} \Rightarrow \begin{cases} I_A(s) = \dfrac{1}{(L_A s + R_A)}[E_i(s) - E_{emf}(s)] \\ \Omega(s) = s\theta(s) = \dfrac{1}{J_A s + B}[T_m(s) - T_L(s)] \\ T_m(s) = K_T I_A(s) \\ E_{emf}(s) = K_b s \theta(s) = K_b \Omega(s) \end{cases}$$

$$\Rightarrow (J_A s + B)\Omega(s) = K_T \cdot \frac{1}{L_A s + R_A}[E_i(s) - K_b \Omega(s)] - T_L(s)$$

$$\Rightarrow (J_A s + B)(L_A s + R_A)\Omega(s) = K_T E_i(s) - \underline{K_T K_b \Omega(s)} - (L_A s + R_A) T_L(s)$$

$$\{J_A L_A s^2 + (L_A B + R_A J_A) s + B R_A + K_T K_b\} \Omega(s) = K_T E_i(s) - (L_A s + R_A) T_L(s)$$

I/O Model from $e_i(t)$ and τ_L to angular speed ω:

$$\ddot{\omega} + \left(\frac{B}{J_A} + \frac{R_A}{L_A}\right)\dot{\omega} + \left(\frac{R_A B}{L_A J_A} + \frac{K_T K_b}{L_A J_A}\right)\omega = \frac{K_T}{L_A J_A} e_i(t) - \frac{(L_A \dot{\tau}_L + R_A \tau_L)}{L_A J_A}$$

I/O Model from $e_i(t)$ and τ_L to angular position θ:

$$\dddot{\theta} + \left(\frac{B}{J_A} + \frac{R_A}{L_A}\right)\ddot{\theta} + \left(\frac{R_A B}{L_A J_A} + \frac{K_T K_b}{L_A J_A}\right)\dot{\theta} = \frac{K_T}{L_A J_A} e_i(t) - \frac{(L_A \dot{\tau}_L + R_A \tau_L)}{L_A J_A}$$

DC Motors - Modeling

Transfer Functions:

$$\Omega(s)=\underbrace{\frac{\frac{K_T}{L_AJ_A}}{s^2+\left(\frac{B}{J_A}+\frac{R_A}{L_A}\right)s+\left(\frac{R_AB}{L_AJ_A}+\frac{K_TK_b}{L_AJ_A}\right)}}_{G_{\Omega E}}\cdot E_i(s)$$

$$+\underbrace{\frac{-\left(\frac{1}{J_A}s+\frac{R_A}{L_AJ_A}\right)}{s^2+\left(\frac{B}{J_A}+\frac{R_A}{L_A}\right)s+\left(\frac{R_AB}{L_AJ_A}+\frac{K_TK_b}{L_AJ_A}\right)}}_{G_{\Omega T}(s)}\cdot T_L(s)$$

$$\Theta(s)=\frac{1}{s}G_{\Omega E}\cdot E_i(s)+\frac{1}{s}G_{\Omega T}\cdot T_L(s)$$

Q: Let the load torque be zero (No Load), what is the steady state speed (No-Load Speed) of the motor for a constant input voltage V ?

$e_i(t) = V = \text{constant}$

$$\Omega(s) = G_{\Omega E}\, E_i(s) = G_{\Omega E}(s)\,\frac{V}{s} \Rightarrow \omega(t) \rightarrow \omega_{ss} = G_{\Omega E}(0)\, V = \frac{K_T}{BR_A + K_TK_b} V$$

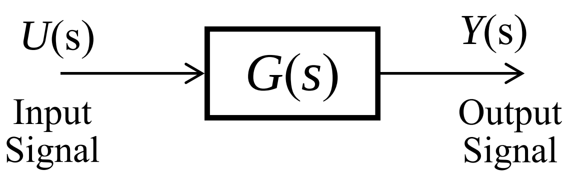
Q: Let the load torque $\tau_L = T$, what is the steady state speed of the motor for a constant input voltage V ?

$$\omega(t) \longrightarrow \omega_{ss} = G_{\Omega E}(0)\, V + G_{\Omega T}(0)\, T = \frac{K_T}{BR_A + K_TK_b} V - \frac{R_A}{R_AB + K_TK_b} T$$

Block Diagram Representation

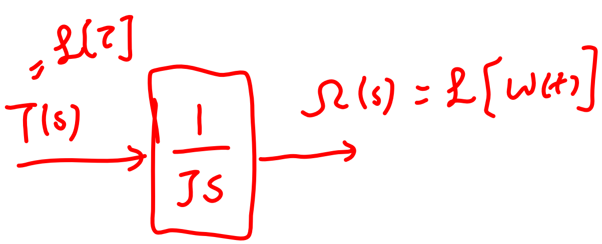
- Differential Equation → Transfer Function (System & Signals)

$$Y(s) = G(s) \cdot U(s)$$



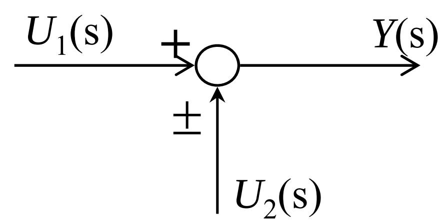
Ex: Draw the block diagram for the following DE:

$$J\dot{\omega} = \tau$$



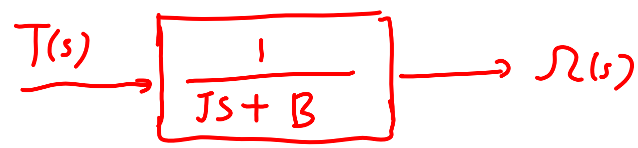
- Signal Addition/Subtraction

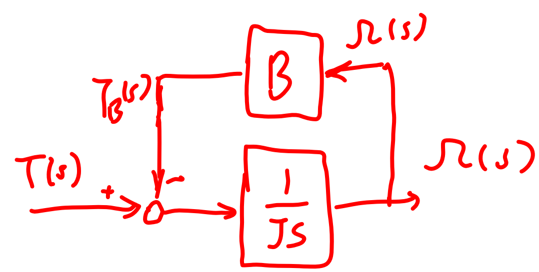
$$Y(s) = U_1(s) \pm U_2(s)$$



Ex: Draw the block diagram for the following DE:

$$J\dot{\omega} + B\omega = \tau$$



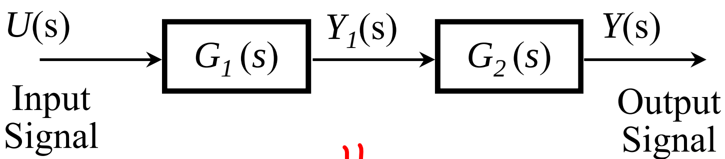


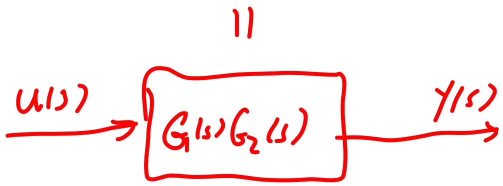
Block Diagram Representation

- Transfer Function in Series

$$Y(s) = G_2(s) \cdot Y_1(s) \ , \ Y_1(s) = G_1(s) \cdot U(s)$$

$$Y(s) = \left[G_2(s) \cdot G_1(s) \right] \cdot U(s)$$



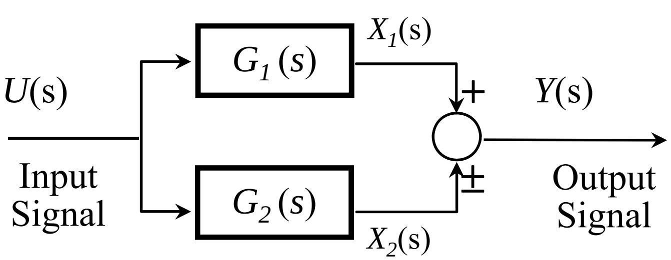


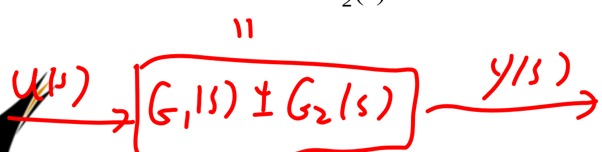
- Transfer Function in Parallel

$$X_1(s) = G_1(s) \cdot U(s) \ , \ X_2(s) = G_2(s) \cdot U(s)$$

$$Y(s) = X_1(s) \pm X_2(s)$$

$$= \left[G_1(s) \pm G_2(s) \right] \cdot U(s)$$



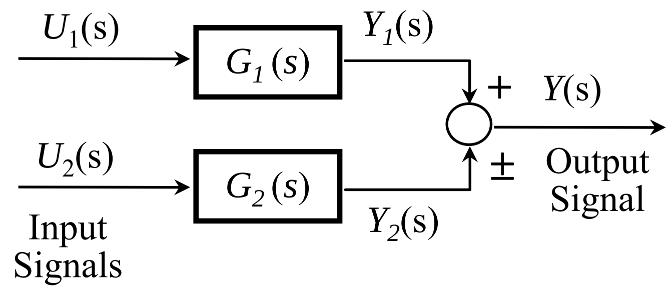


- Multiple Inputs

$$Y_1(s) = G_1(s) \cdot U_1(s) \ , \quad Y_2(s) = G_2(s) \cdot U_2(s)$$

$$Y(s) = Y_1(s) \pm Y_2(s)$$

$$= G_1(s) \cdot U_1(s) \pm G_2(s) \cdot U_2(s)$$

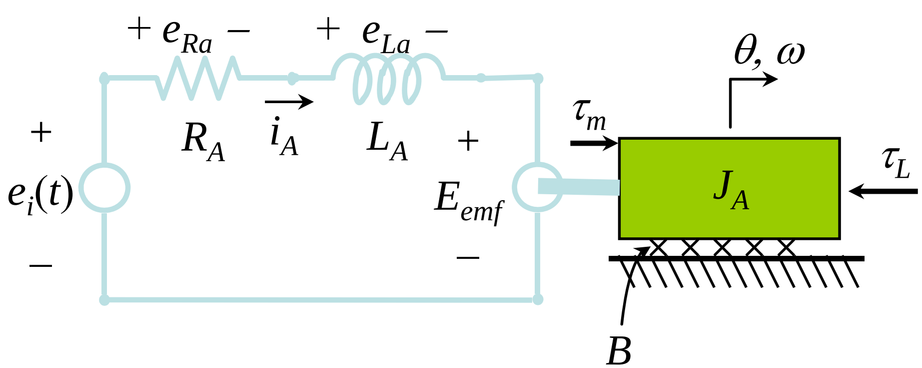


Ex: Draw the block diagram for

$$L \frac{d}{dt} i + R i + K\omega = e_i$$

Block Diagram Representation of DC Motors

Schematic



EOM in s-Domain:

Governing Equations:

$$L_A \frac{d}{dt} i_A + R_A i_A + E_{emf} = e_i(t) \quad \text{(Electrical)}$$

$$\Rightarrow I_A(s) = \frac{1}{L_A s + R_A}\left[E_i(s) - E_{emf}(s)\right]$$

$$J_A \dot{\omega} + B\omega = \tau_m - \tau_L \quad \text{(Mechanical)}$$

$$\Omega(s) = \frac{1}{J_A s + B}\left[T_m(s) - T_L(s)\right]$$

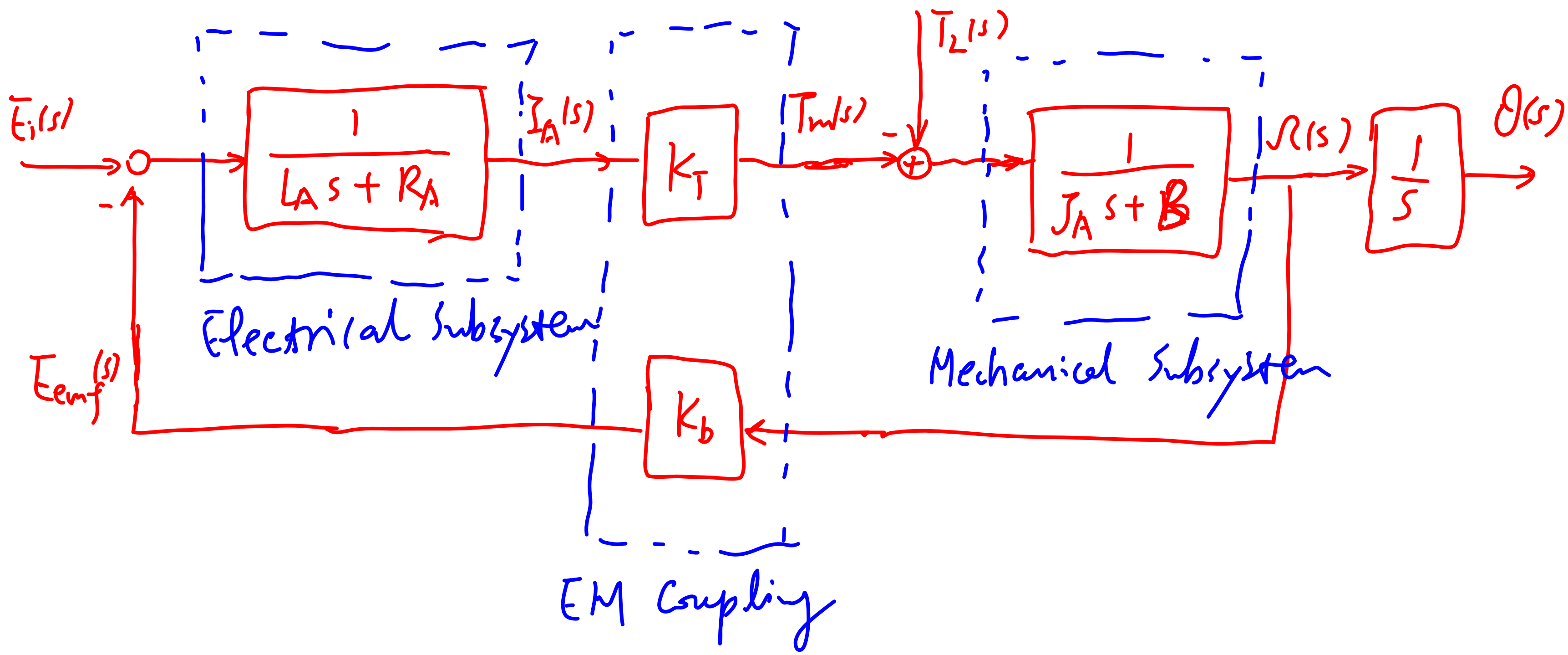
$$\left.\begin{aligned} \tau_m &= K_T \cdot i_A \\ E_{emf} &= K_b \cdot \omega \end{aligned}\right\} \quad \text{(EM Coupling)}$$

$$T_m(s) = K_T I_A(s)$$

$$E_{emf}(s) = K_b \Omega(s)$$

PURDUE MECHANICAL ENGINEERING

Block Diagram Representation of DC Motors

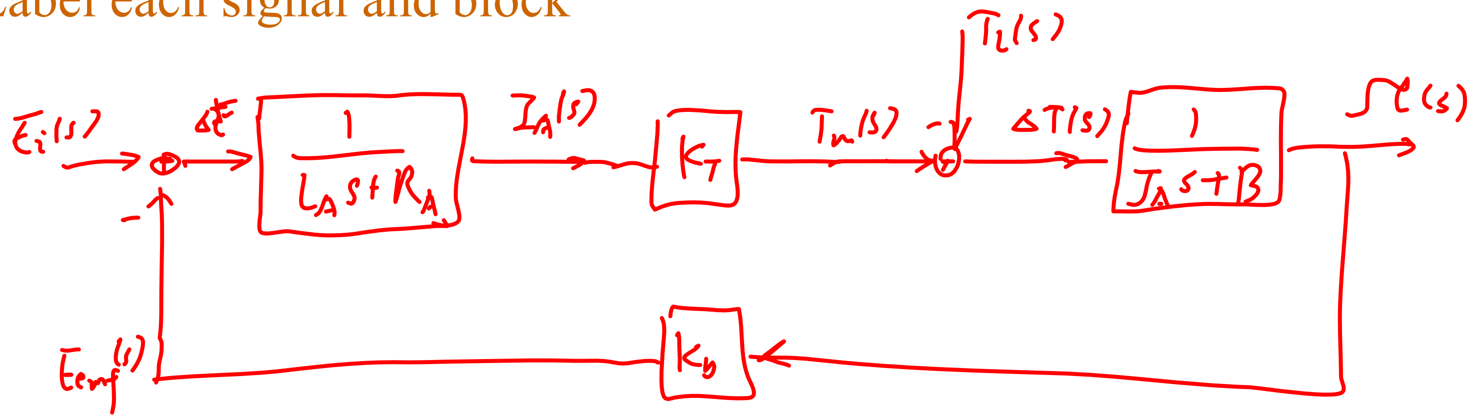


Q: Now that we generated a block diagram of a voltage driven DC Motor, can we derive the transfer function of this motor from its block diagram ? (This is the same as asking you to reduce the multi-block diagram to a simpler form just relating inputs $e_i(t)$ and τ_L to the output, either ω or θ)

Block Diagram Reduction

From Block Diagram to Transfer Function

- Label each signal and block



- Write down the relationships among signals based on the block diagram

$$
\begin{cases}
I_A(s) = \dfrac{1}{L_A s + R_A} \Delta E(s) \\
\Delta E(s) = E_i(s) - E_{emf}(s) \\
E_{emf}(s) = K_b \, \Omega(s) \\
\Omega(s) = \dfrac{1}{J_A s + B} \Delta T(s) \\
\Delta T(s) = T_m(s) - T_L(s) \\
T_m(s) = K_T \, I_A(s)
\end{cases}
$$

$$6 \text{ eqs} = \overbrace{I_A(s), \Delta E(s), E_{emf}(s), \underset{\overset{\parallel}{Y(s)}}{\Omega(s)}, \Delta T, T_m}^{\text{UV}}$$

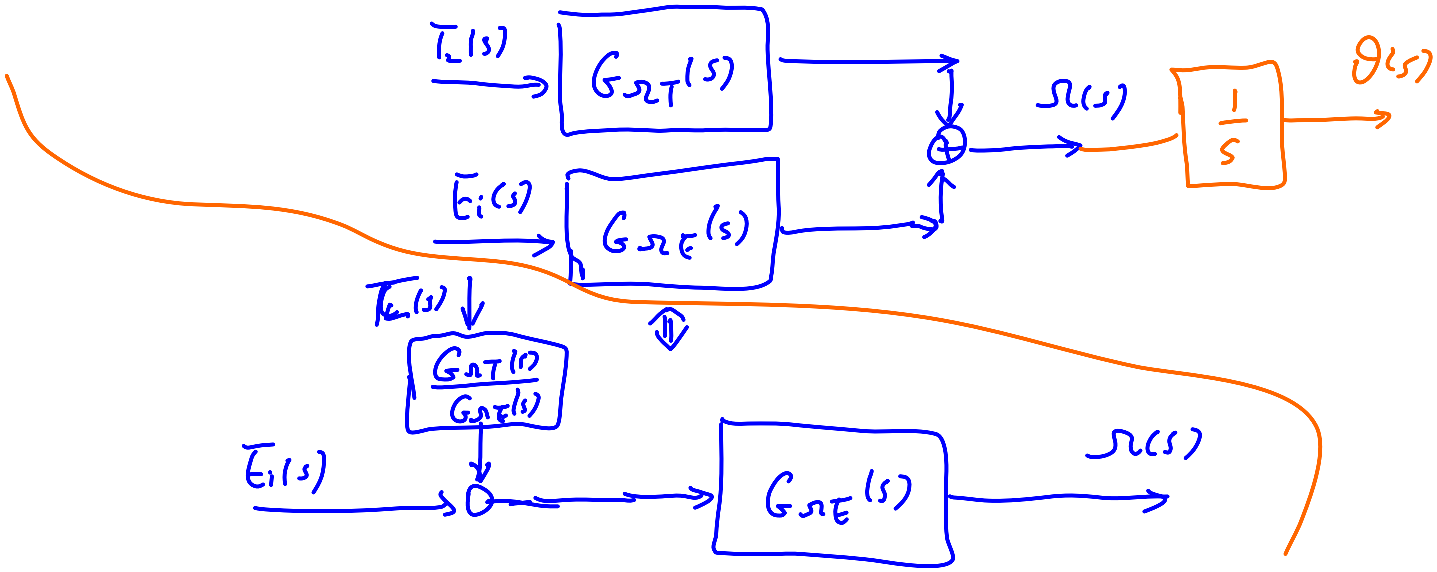
Two inputs:

$E_i(s)$; $T_L(s)$

Block Diagram Reduction

- Solve for the output signal in terms of the input signals

$$\Omega(s) = G_{\Omega E}(s)\, E_i(s) + G_{\Omega T}(s)\, T_L(s) = G_{\Omega E}(s)\left[E_i(s) + \frac{G_{\Omega T}(s)}{G_{\Omega E}(s)}\, T_L(s)\right]$$



- Substitute the transfer functions' label with the actual formula and simplify

$$\Omega(s) = \underbrace{\frac{\dfrac{K_T}{L_A J_A}}{s^2 + \left(\dfrac{B}{J_A} + \dfrac{R_A}{L_A}\right)s + \left(\dfrac{R_A B}{L_A J_A} + \dfrac{K_T K_b}{L_A J_A}\right)}}_{G_{\Omega E}} \cdot E_i(s) - \underbrace{\frac{\dfrac{1}{J_A}s + \dfrac{R_A}{L_A J_A}}{s^2 + \left(\dfrac{B}{J_A} + \dfrac{R_A}{L_A}\right)s + \left(\dfrac{R_A B}{L_A J_A} + \dfrac{K_T K_b}{L_A J_A}\right)}}_{G_{\Omega T}} \cdot T_L(s)$$

$$\Theta(s) = \frac{1}{s} G_{\Omega E} \cdot E_i(s) + \frac{1}{s} G_{\Omega T} \cdot T_L(s)$$

Example

(A) Given the following specification of a DC motor and assume there is no load, find its transfer function from input voltage to motor angular speed

$L_A = 2$ mH $= 2\times10^{-3}$ H

$R_A = 10\ \Omega$

$K_T = 0.06$ Nm/A

$J_A = 5 \times 10^{-6}$ Kg m^2

$B = 3 \times 10^{-6}$ Nm/(rad/sec)

$$G_{\Omega E}(s) = \frac{6\times10^{6}}{s^2 + 5000s + 3.63\times10^{5}}$$

(B) Find the poles of the transfer function.

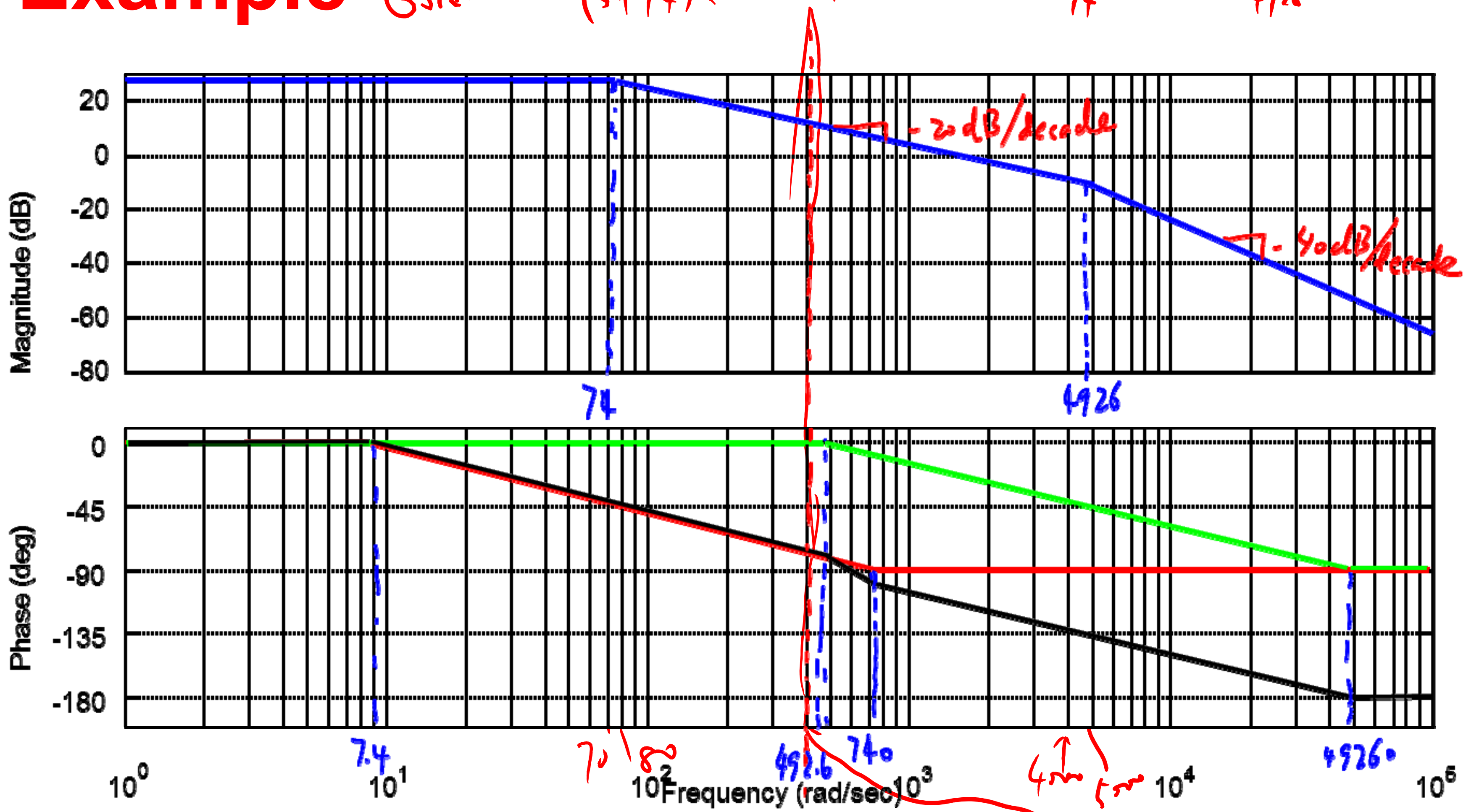
$$s^2 + 5000s + 3.63\times10^{5} = 0$$

$$\Rightarrow \begin{cases} p_1 = -74 \\ p_2 = -4926 \end{cases}$$

(C) Plot the Bode diagram of the transfer function

Example

$$G_{NE}(s) = \frac{6\times10^6}{(s+74)(s+4926)} = 16.53 \times \frac{1}{\frac{1}{74}s+1} \times \frac{1}{\frac{1}{4926}s+1}$$



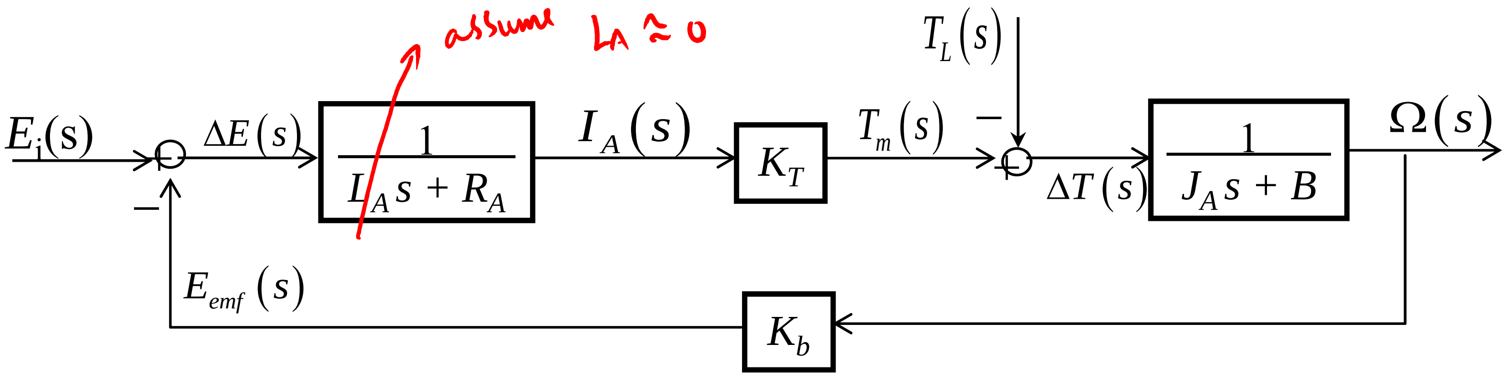
Q: If we are only interested in the system response up to 400 rad/sec, can we simplify our model ? How would you simplify the model ?

$$G_{NE}(s) \approx 16.53 \times \frac{1}{\frac{1}{74}s+1}$$

Model Reduction

- **Neglect Electrical Dynamics**

Derive the model for the DC motor, if the armature inductance L_A is neglected:



$$\Omega(s) = \frac{\frac{K_T}{L_A J_A}}{s^2 + \left(\frac{B}{J_A} + \frac{R_A}{L_A}\right)s + \left(\frac{R_A B}{L_A J_A} + \frac{K_T K_b}{L_A J_A}\right)} \cdot E_i(s) - \frac{\frac{1}{J_A}s + \frac{R_A}{L_A J_A}}{s^2 + \left(\frac{B}{J_A} + \frac{R_A}{L_A}\right)s + \left(\frac{R_A B}{L_A J_A} + \frac{K_T K_b}{L_A J_A}\right)} \cdot T_L(s)$$

$$= \frac{K_T}{\underbrace{L_A J_A}_{0} s^2 + (B L_A + R_A J_A)s + (R_A B + K_T K_b)} E_i(s) - \frac{\underbrace{L_A s}_{0} + R_A}{\cdots} T_L(s)$$

$$= \frac{K_T}{R_A J_A s + (R_A B + K_T K_b)} E_i(s) - \frac{R_A}{\cdots} T_L(s)$$

By neglecting the effect of armature inductance, we reduced the order of the model from two to one when the output is the motor speed !

Model Reduction

Q: Physically, what do we mean by neglecting armature inductance?

By neglecting the armature inductance, we are assuming that it takes no time for the current to reach its steady state value when there is a step change in input voltage, i.e. a sudden change in input voltage will result in a sudden change in the armature current, which in turns will result in a sudden change in the motor torque output. This is equivalent to have direct control of the motor current.

Mathematically:

$$L_A \frac{d}{dt} i_A + R_A i_A + E_{emf} = e_i(t)$$

$$J_A \dot{\omega} + B\omega = \tau_m - \tau_L$$

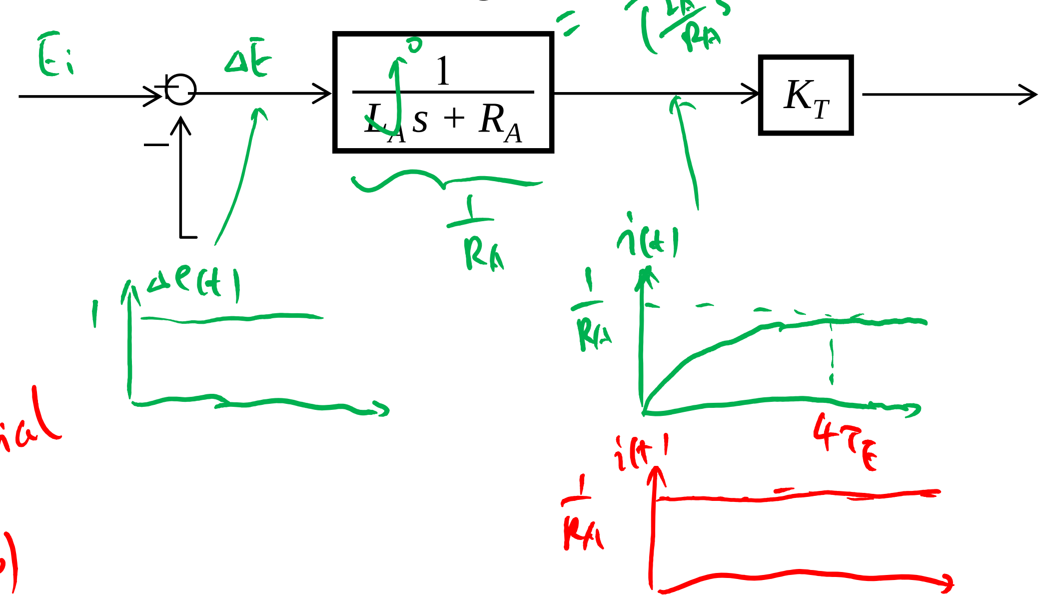
$$\begin{cases} \tau_m = K_T \cdot i_A \\ E_{emf} = K_b \cdot \omega \end{cases}$$

Reality

Ignore electrical dynamics ($L_A \approx 0$)

From Block Diagram:

$$= \frac{1/R_A}{\left(\frac{L_A}{R_A}s+1\right)} = \frac{1/R_A}{\tau_E s+1}$$

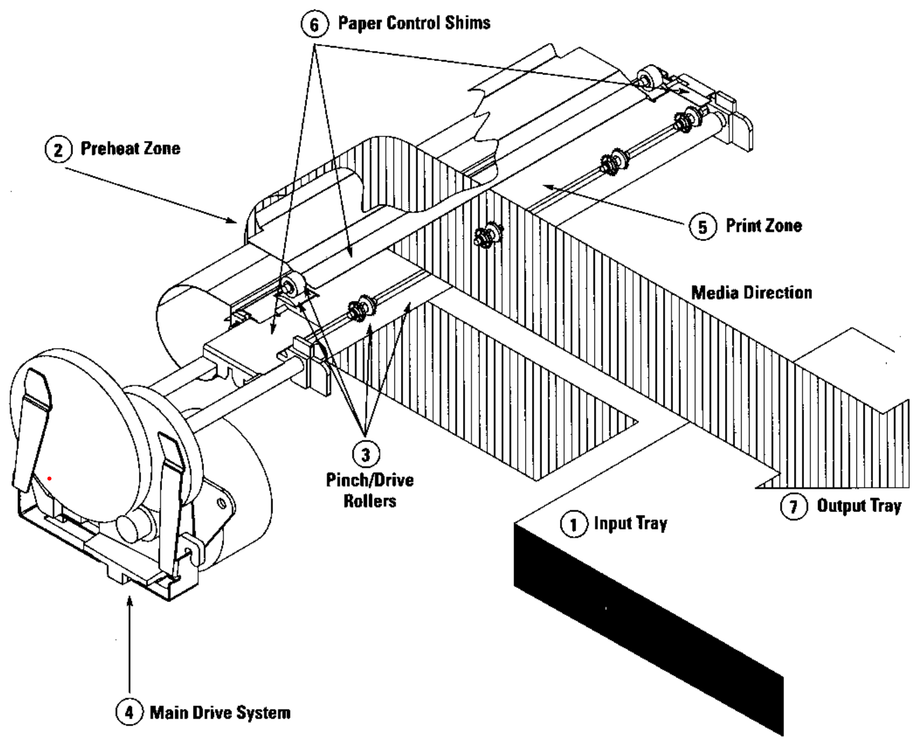


Q: Practically, when can we neglect electrical dynamics when modeling motors?

Example

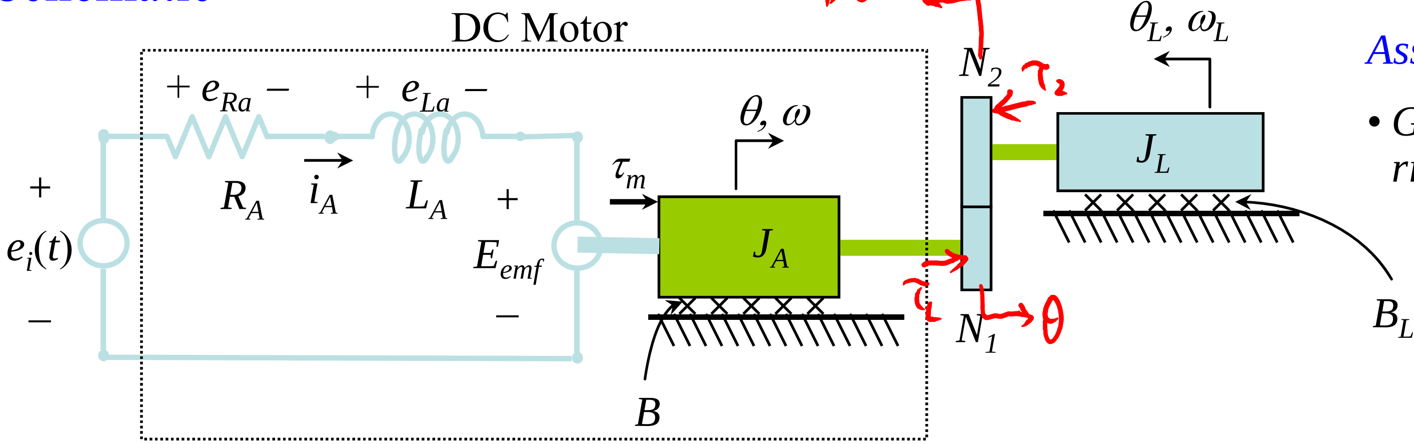
Media Advance System in InkJet Printers

The figure on the right shows the media advance system of a typical inkjet printer. The objective of the system is to precisely and quickly position the media such that ink droplets can be precisely "dropped" on to the media to form "nice looking" images. The system is driven by a DC motor through two sets of gear trains. You, the "new kid" on the development team is given the task of specifying a motor and design the control system that will achieve the desirable performance. Sometime your manager will also walk by your desk and ask you if certain performance is achievable. How would you start your first engineering project?



Example

Schematic



Assumptions:

- *Gears and shafts are rigid and massless.*

Gear ratio:

$$n = \frac{N_2}{N_1}$$

Block diagram of the load inertia:

$$J_L\ddot{\theta}_L = -\tau_2 - B_L\dot{\theta}_L$$

$$\Rightarrow \quad J_Ls^2\theta_L(s) = -\tau_2(s) - B_Ls\,\theta_L(s)$$

$$\Rightarrow \quad (J_Ls^2 + B_Ls)\,\theta_L(s) = -\tau_2(s)$$

$$-\tau_2(s) \longrightarrow \boxed{\frac{1}{J_Ls^2 + B_Ls}} \longrightarrow \theta_L(s)$$

or

$$-\tau_2(s) \longleftarrow \boxed{J_Ls^2 + B_Ls} \longleftarrow \theta_L(s)$$

Block diagram of the gear train:

$$\theta = n\theta_L \quad \text{or} \quad \theta_L = \frac{1}{n}\theta$$

$$\tau_L = -\frac{1}{n}\tau_2 \quad \text{or} \quad \tau_2 = -n\tau_L$$

$$\tau_L \longleftarrow \boxed{\frac{1}{n}} \longleftarrow -\tau_2(s) \qquad \tau_L \longrightarrow \boxed{n} \longrightarrow -\tau_2$$

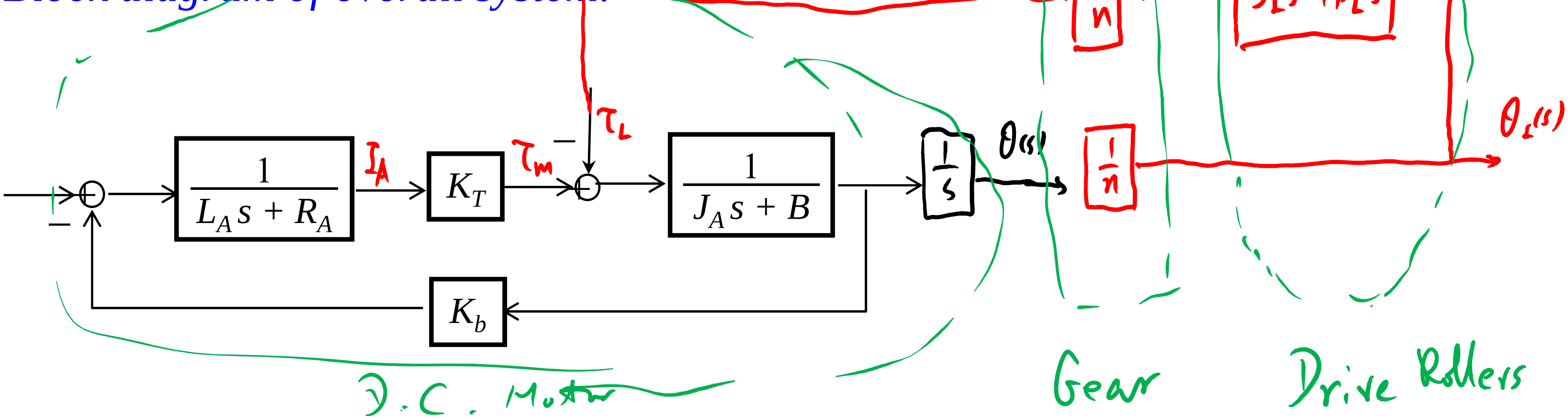
or

$$\theta(s) \longrightarrow \boxed{\frac{1}{n}} \longrightarrow \theta_L(s) \qquad \theta \longleftarrow \boxed{n} \longleftarrow \theta_L$$

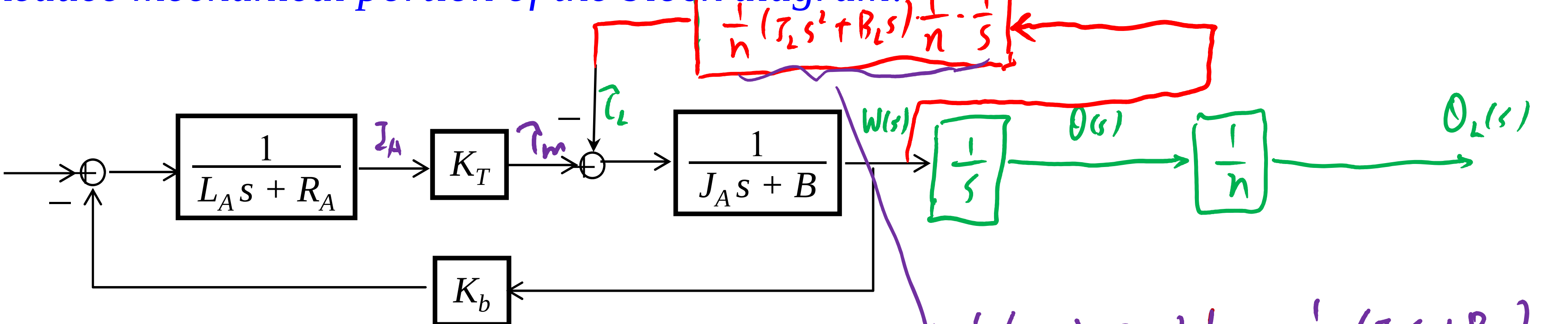
PURDUE MECHANICAL ENGINEERING

Example

Block diagram of overall system:



Reduce mechanical portion of the block diagram:



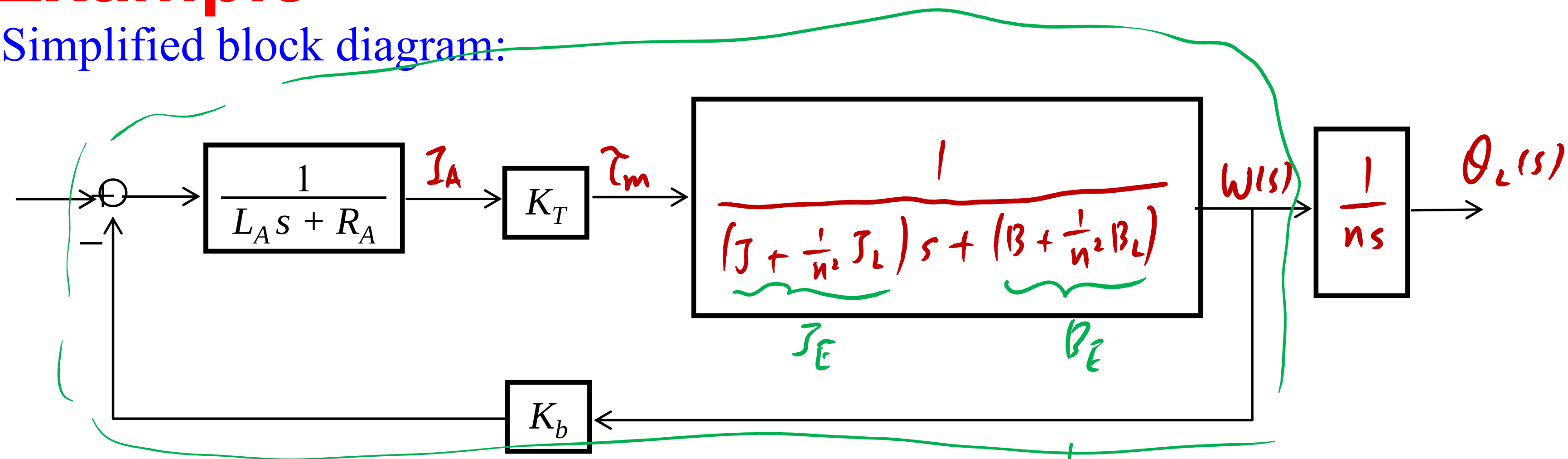
$$\frac{1}{n^2}(J_L s^2 + B_L s)\frac{1}{s} = \frac{1}{n^2}(J_L s + B_L)$$

$$\Rightarrow W(s) = \frac{1}{J_A s + B}(\tau_m - \tau_L) = \frac{1}{J_A s + B}\left[\tau_m - \frac{1}{n^2}(J_L s + B_L)\, W(s)\right]$$

$$\Rightarrow W(s) = \frac{1}{\left(J_A + \frac{1}{n^2}J_L\right)s + \left(B + \frac{1}{n^2}B_L\right)}\tau_m$$

Example

Simplified block diagram:



Transfer Function from input voltage $E_i(s)$ to angular position of the load $\theta(s)$:

$$G_{E_i\theta}(s) = \frac{\theta_L(s)}{E_i(s)} = \frac{1}{ns} \cdot \frac{K_T}{L_A J_E s^2 + (B_E L_A + R_A J_E)s + (R_A B_E + K_b K_T)}$$

Q: Is this system stable ?

Not stable since there is one pole at origin

Q: What command (voltage) would you use to move the roller's angular position by, say 60° ?