Rotational Mechanical Systems

• Variables
• Basic Modeling Elements
• Interconnection Laws
• Derive Equation of Motion (EOM)
Variables

- \( \theta \) : angular displacement [rad]
- \( \omega \) : angular velocity [rad/sec]
- \( \alpha \) : angular acceleration [rad/sec^2]
- \( \tau \) : torque [Nm]
- \( p \) : power [Nm/sec]
- \( w \) : work (energy) [Nm]
  
  1 [Nm] = 1 [J] (Joule)

\[
\frac{d}{dt} \theta = \dot{\theta} = \omega
\]

\[
\ddot{\omega} = \frac{d}{dt} \left( \frac{d}{dt} \theta \right) = \frac{d^2}{dt^2} \theta = \ddot{\theta} = \alpha
\]

\[
p = \tau \cdot \omega = \tau \cdot \dot{\theta} = \frac{d}{dt} w
\]

\[
w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t) \, dt
\]

\[
= w(t_0) + \int_{t_0}^{t_1} (\tau \cdot \dot{\theta}) \, dt
\]
Basic Rotational Modeling Elements

• **Spring**
  - Stiffness Element
  - Stores Potential Energy.
  - e.g., shafts
  \[ \tau_S = K(\theta_2 - \theta_1) \]

• **Damper**
  - Friction Element
  - Dissipate Energy.
  - e.g., bearings, bushings, ...
  \[ \tau_D = B(\dot{\theta}_2 - \dot{\theta}_1) = B(\omega_2 - \omega_1) \]
Basic Rotational Modeling Elements

• **Moment of Inertia**
  - Inertia Element

  \[ J = \frac{1}{12} ML^2 \]  
  \( (a) \)

  \[ J = \frac{1}{2} MR^2 \]  
  \( (b) \)

  \[ J = J_O + Mr^2 \]

  \[ J = J_O + Mr^2 = \frac{1}{2} MR^2 + Md^2 \]

• **Parallel Axis Theorem**

  \[ J = J_O + Mr^2 \]

  \[ J = J_O + Mr^2 = \frac{1}{12} ML^2 + M\left(\frac{L}{2}\right)^2 \]

  \[ J = J_O + Mr^2 = \frac{1}{3} ML^2 \]

  \[ J = J_O + Mr^2 \]
Interconnection Laws

- **Newton’s Second Law**
  \[
  \frac{d}{dt} (J \omega) = J \ddot{\theta} = \sum_i \tau_{EXTi}
  \]

- **Newton’s Third Law**
  - Action & Reaction Torque; equal but have opposite signs

- **Angular Displacement Law**
  - Elements connecting to the same location have the same angular displacements
Example

Derive a model (EOM) for the following system:

**FBD:** (Use Right-Hand Rule to determine direction)

\[
\begin{align*}
J \ddot{\theta} &= \sum \tau = \tau + \tau_s + \tau_D \\
\tau_s &= K (0 - \theta) = -K \theta \\
\tau_D &= B (0 - \dot{\theta}) = -B \dot{\theta} \\
J \ddot{\theta} &= \tau - K \theta - B \dot{\theta}
\end{align*}
\]

\[
J \ddot{\theta} + B \dot{\theta} + K \theta = \tau
\]
Energy Distribution

- EOM of a simple Mass-Spring-Damper System

\[ J\ddot{\theta} + B\dot{\theta} + K\theta = \tau(t) \]

**Contribution of Inertia**

**Contribution of the Damper**

**Contribution of the Spring**

**Total Applied Torque**

*We want to look at the energy distribution of the system. How should we start?*

- Multiply the above equation by angular velocity term \( \omega \): \( \Leftarrow \) *What have we done?*

\[ J\ddot{\theta} \cdot \dot{\theta} + B\dot{\theta} \cdot \dot{\theta} + K\theta \cdot \dot{\theta} = \tau(t) \cdot \omega \]

- Integrate the second equation w.r.t. time: \( \Leftarrow \) *What are we doing now?*

\[
\int_{t_0}^{t_1} J \ddot{\theta} \cdot \dot{\theta} \, dt + \int_{t_0}^{t_1} B \omega \cdot \omega \, dt + \int_{t_0}^{t_1} K \theta \cdot \dot{\theta} \, dt = \int_{t_0}^{t_1} \tau(t) \cdot \omega \, dt
\]

- \( \Delta KE \) for change of kinetic energy
- \( \Delta PE \) for change of potential energy
- energy dissipated by damper

*Total work done by the applied torque \( \tau(t) \) from time \( t_0 \) to \( t_1 \)*
Motion Transfer Elements

• Lever

(Motion Transformer Element)

Assume massless and no joint friction torque

\[ \frac{x_1}{d_1} = \sin \theta = \frac{x_2}{d_2} \quad \Rightarrow \quad \frac{x_1}{x_2} = \frac{d_1}{d_2} = N \]

\[ \theta = \int \tau = \sum \tau = -f_1 d_1 \omega \theta + f_2 d_2 \omega \theta \]

\[ \Rightarrow \quad \frac{f_2}{f_1} = \frac{d_1}{d_2} = N \]

Energy Conservation

\[ f_1 x_1 = f_2 x_2 \]

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Motion Transfer Elements

- **Ideal Gears**

  Gear 1
  \[ 0 = J_1 \ddot{\theta}_1 = \sum \tau = \tau_1 - f_{12}r_1 \]
  Gear 2
  \[ 0 = J_2 \ddot{\theta}_2 = \sum \tau = -\tau_2 + f_{21}r_2 \]

  \( \begin{align*}
  \tau_1 &= f_{12}r_1 \\
  \tau_2 &= f_{21}r_2
  \end{align*} \)

  \( \theta_1 = \frac{r_1}{N} \)
  \( \theta_2 = \frac{r_2}{N} \)

  \( \omega_1 = \frac{r_1}{N} \)
  \( \omega_2 = \frac{r_2}{N} \)

- **Rack and Pinion**

  Ideal case (negligible inertias and damping)
  \[ 0 = I_{o-cm} \ddot{\theta} = \tau - F_{gr}r \]
  \[ 0 = M_r \ddot{x} = F_{gr} - F \]

  \[ \frac{F}{r} = \frac{1}{\tau} \]
  \[ \frac{\dot{x}}{r} = \frac{1}{\dot{\theta}} \]

  \( N = \frac{1}{r} \)

negligible moment of inertia and friction torques
Example (coupled translation-rotation)

Real world (non-negligible inertias and/or damping)

Think about performance of lever and gear train in real life
Example

- Rolling without slipping

Elemental Laws:

\[ M\ddot{x} = f - f_s - f_D - f_c \]
\[ 0 = Mg - N\dot{\theta} \]
\[ \frac{1}{2}MR^2\ddot{\theta} = f_c \cdot R \]
\[ f_s = K(x + \theta) \]
\[ f_D = B(\dot{x} - \dot{\theta}) \]

UV:
\[ x, f_s, f_D, f_c, N, \theta \]

Roll without slipping:
\[ \dot{x} - R\ddot{\theta} = 0 \iff x = R\dot{\theta} \]

\[ \Rightarrow f_c = \frac{1}{2}MR\ddot{\theta} = \frac{1}{2}MR\left(\frac{\dddot{x}}{R}\right) = \frac{1}{2}M\dddot{x} \]
Example (cont.)

I/O Model (Input: \( f \), Output: \( x \)):

\[
M \ddot{x} = f - kx - B \dot{x} - \frac{1}{2} M \dddot{x}
\]

\[
\Rightarrow \frac{3}{2} M \dddot{x} + B \ddot{x} + kx = f
\]

Q: How would you decide whether or not the disk will slip?

\[
|f_c| < \mu_s N = \mu_s Mg
\]

\[
\Rightarrow \left| \frac{1}{2} M \dddot{x} \right| < \mu_s Mg
\]

Q: How will the model be different if the disk rolls and slips?

\( x \neq \theta \)

\[
f_c = \mu_s Mg
\]
General Mechanical Systems

• Example

Derive the differential equation of motions (EOMs) for the system in terms of the outputs $x$ and $\theta$, and the input $u(t)$. 

\[ l_0 = \frac{1}{2} mR^2 \]

FBD:
Example (cont.)

Elemental Equations:

\[ f_{s_1} = k (x_w - u) \]

\[ m \ddot{x}_w = -f_{s_1} + f_c + f_{s_2} + f_f \]

\[ I_o \ddot{\theta} = -f_f \cdot R \]

\[ f_c = C (\dot{x} - \dot{x}_w) \]

\[ f_{s_2} = k (x - x_w) \]

\[ m \ddot{x} = f_{s_2} - f_{s_2} - f_c \]

\[ f_{s_3} = k (-x - o) \]

No slip: \( x_w = R \theta \)

UVs: \( f_{s_1}, x_w, f_c, f_{s_2}, f_f, \theta, x, f_{s_3} \)

8 UVs = 8 eqs \( \Rightarrow \) have all the equations

Matrix Form:

\[
\begin{bmatrix}
m & 0 & \frac{I_o}{R} \\
0 & m R \frac{I_o}{R} & 0 \\
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{\theta}
\end{bmatrix}
+
\begin{bmatrix}
-C & -C R \\
-C & C R
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{\theta}
\end{bmatrix}
+
\begin{bmatrix}
2k & -k R \\
-k & 2k R
\end{bmatrix}
\begin{bmatrix}
x \\
\theta
\end{bmatrix}
= \begin{bmatrix}
0 \\
k u
\end{bmatrix}
\]
EXAMPLE - INVETED PENDULUM ON A CART

Elemental Equations:

\[
M\ddot{x} = F - N - F_c - b_x \dot{x} \\
0 = p - Mg + Ng \\
m \ddot{x}_c = N \\
m \ddot{y}_c = -mg - p \\
I \ddot{\theta} = -NL\omega \dot{\theta} - PL\dot{\theta}^2 - b_o \dot{\theta} \\
\]
INVETED PENDULUM ON A CART

- Linearization assuming small angle and $b = 0$

  If $\theta \ll 1$, $\cos \theta \approx 1$, $\sin \theta \approx \theta$, and neglect all higher-order terms:

\[
\begin{align*}
(M+m) \ddot{x} + mL \ddot{\theta} + bx \dot{x} &= F - F_c \\
ml \ddot{x} + (1 + mL^2) \dot{\theta} + b_\theta \dot{\theta} - mgL\dot{\theta} &= 0
\end{align*}
\]

Matrix Form:

\[
\begin{bmatrix}
M+m & mL \\
ml & I+ml^2
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
bx & 0 \\
0 & b_\theta
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{\theta}
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 \\
0 & -mgL
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{\theta}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0
\end{bmatrix}\left(F - F_c\right)
\]
EXAMPLE – CONTINUOUS ROLLING MILL

FBD:

Suppose that the motion of the adjustable roller has a damping coefficient $b$, and that the force exerted by the rolled material on the adjustable roller is proportional to the material's change in thickness: $F_x = c(T - x)$. Suppose further that the rack-and-pinion has an effective radius of $R$ and the DC motor can be modeled by a torque input source of $\tau_m$ for this problem.

**Inputs:**

**Output:**
Example (cont.)

Elemental Equations:

Input-output Form: