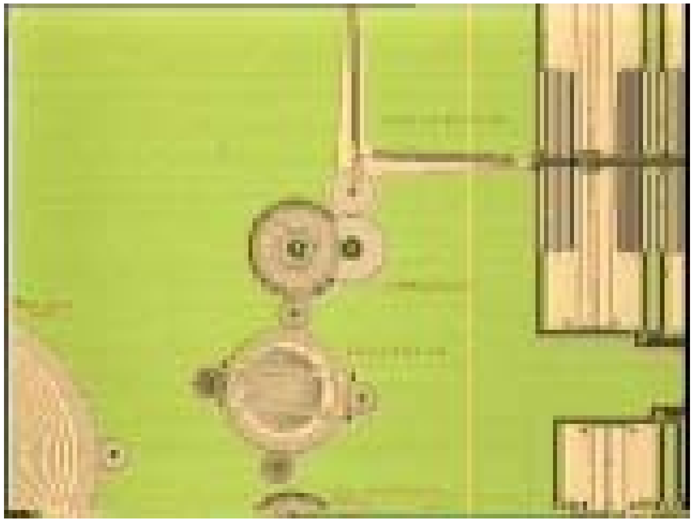
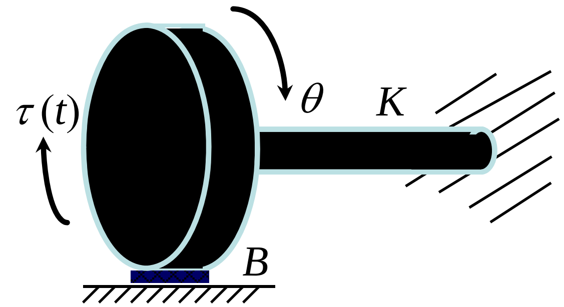
Rotational Mechanical Systems

- **Variables**
- **Basic Modeling Elements**
- **Interconnection Laws**
- **Derive Equation of Motion (EOM)**



Variables



- $\boldsymbol{\theta}$: angular displacement **[rad]**
- $\boldsymbol{\omega}$: angular velocity **[rad/sec]**
- $\boldsymbol{\alpha}$: angular acceleration **[rad/sec²]**
- $\boldsymbol{\tau}$: torque **[Nm]**
- $\boldsymbol{p}$: power **[Nm/sec]**
- $\boldsymbol{w}$: work (energy) **[Nm]**

  1 [Nm] = 1 [J] (Joule)

$$\frac{d}{dt}\theta = \dot{\theta} = \omega$$

$$\dot{\omega} = \frac{d}{dt}\left(\frac{d}{dt}\theta\right) = \frac{d^2}{dt^2}\theta = \ddot{\theta} = \alpha$$

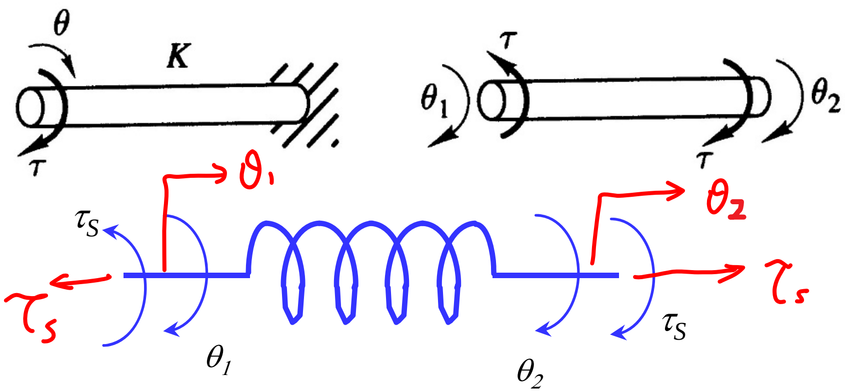
$$p = \tau \cdot \omega = \tau \cdot \dot{\theta} = \frac{d}{dt}w$$

$$w(t_1) = w(t_0) + \int_{t_0}^{t_1} p(t)\,dt$$

$$= w(t_0) + \int_{t_0}^{t_1} (\tau \cdot \dot{\theta})\,dt$$

Basic Rotational Modeling Elements

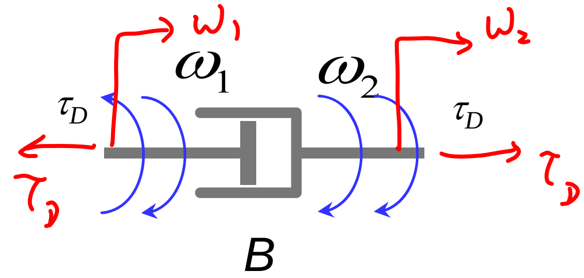
- **Spring**
  - Stiffness Element



$$\tau_S = K(\theta_2 - \theta_1)$$

  - Analogous to Translational Spring.
  - Stores Potential Energy.
  - e.g., shafts

- **Damper**
  - Friction Element



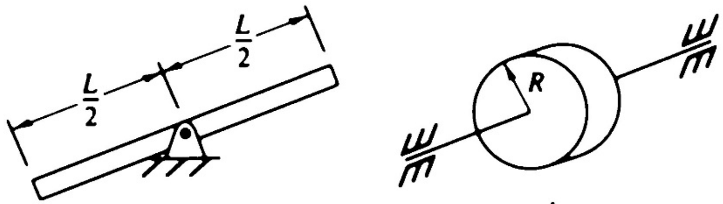
$$\tau_D = B(\dot{\theta}_2 - \dot{\theta}_1) = B(\omega_2 - \omega_1)$$

  - Analogous to Translational Damper.
  - Dissipate Energy.
  - e.g., bearings, bushings, ...

PURDUE MECHANICAL ENGINEERING

Basic Rotational Modeling Elements

- **Moment of Inertia**
  - Inertia Element



$J = \frac{1}{12}ML^2$ (a)

$J = \frac{1}{2}MR^2$ (b)

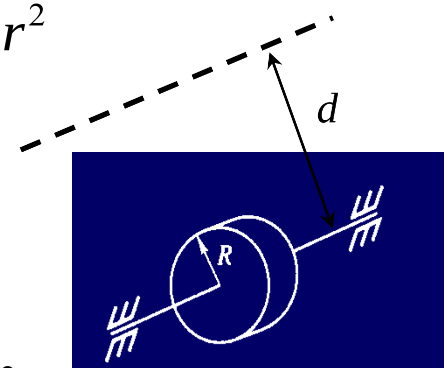
$$J\ddot{\theta} = \sum_i \tau_i$$

  - Analogous to Mass in Translational Motion.
  - Stores Kinetic Energy.

- **Parallel Axis Theorem**

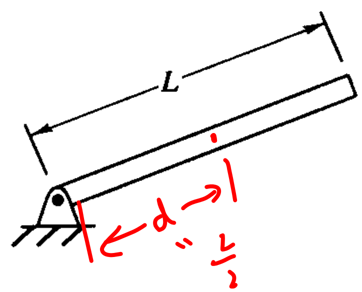
$$J = J_O + Mr^2$$

Ex:



$$J = J_O + M r^2$$

$$= \frac{1}{2}MR^2 + Md^2$$



$$J = J_O + M r^2$$

$$= \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2$$

$$= \frac{1}{3}ML^2$$

Interconnection Laws

- **Newton's Second Law**

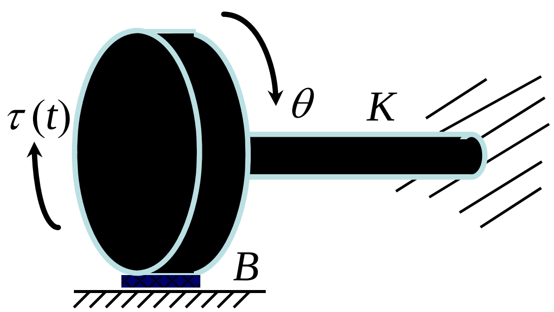
$$\frac{d}{dt}\underbrace{(J\omega)}_{\substack{\text{Angular}\\ \text{Momentum}}} = J\ddot{\theta} = \sum_i \tau_{EXTi}$$

- **Newton's Third Law**
  - Action & Reaction Torque; *equal but have opposite signs*

- **Angular Displacement Law**
  - Elements connecting to the same location have the same angular displacements

Example

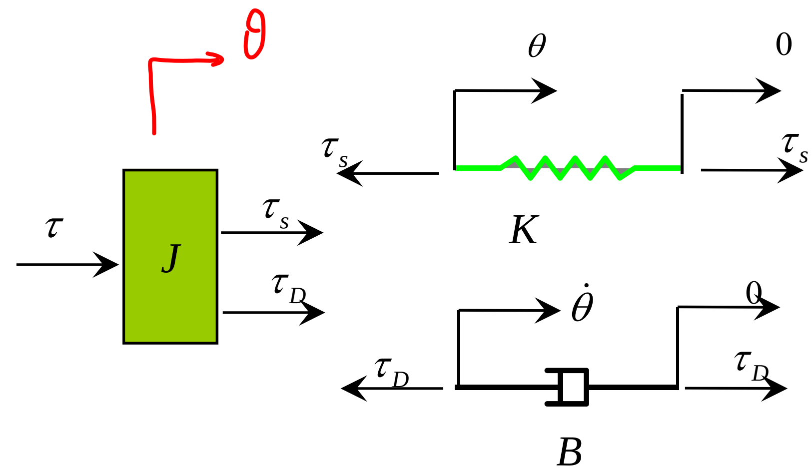
Derive a model (EOM) for the following system:



Translational

$\Rightarrow$

Equivalent



FBD: (Use ***Right-Hand Rule***
to determine direction)

$$J\ddot{\theta} = \sum \tau = \tau + \tau_s + \tau_D$$

$$\tau_s = K(0-\theta) = -K\theta$$

$$\tau_D = B(0-\dot{\theta}) = -B\dot{\theta}$$

$$J\ddot{\theta} = \tau - K\theta - B\dot{\theta}$$

$$J\ddot{\theta} + B\dot{\theta} + K\theta = \tau$$

Energy Distribution

- EOM of a simple Mass-Spring-Damper System

$$\underbrace{J\ddot{\theta}}_{\substack{\text{Contribution}\\\text{of Inertia}}} + \underbrace{B\dot{\theta}}_{\substack{\text{Contribution}\\\text{of the Damper}}} + \underbrace{K\theta}_{\substack{\text{Contribution}\\\text{of the Spring}}} = \underbrace{\tau(t)}_{\substack{\text{Total}\\\text{Applied Torque}}}$$

We want to look at the energy distribution of the system. How should we start ?

- Multiply the above equation by angular velocity term ω : $\Leftarrow$ *What have we done ?*

$$J\ddot{\theta}\cdot\dot{\theta} + B\dot{\theta}\cdot\dot{\theta} + K\theta\cdot\dot{\theta} = \tau(t)\cdot\omega$$

- Integrate the second equation w.r.t. time: $\Leftarrow$ ***What are we doing now ?***

$$\underbrace{\int_{t_0}^{t_1} J\ddot{\theta}\cdot\dot{\theta}\,dt}_{\Delta KE} + \underbrace{\int_{t_0}^{t_1} B\omega\cdot\omega\,dt}_{\int_{t_0}^{t_1} B\omega^2 dt \,\geq\, 0} + \underbrace{\int_{t_0}^{t_1} K\theta\cdot\dot{\theta}\,dt}_{\Delta PE} = \underbrace{\int_{t_0}^{t_1} \tau(t)\cdot\omega\,dt}_{W}$$

$\Delta KE \Downarrow$ change of kinetic energy

$\int_{t_0}^{t_1} B\omega^2 dt \geq 0 \Downarrow$ energy dissipated by damper

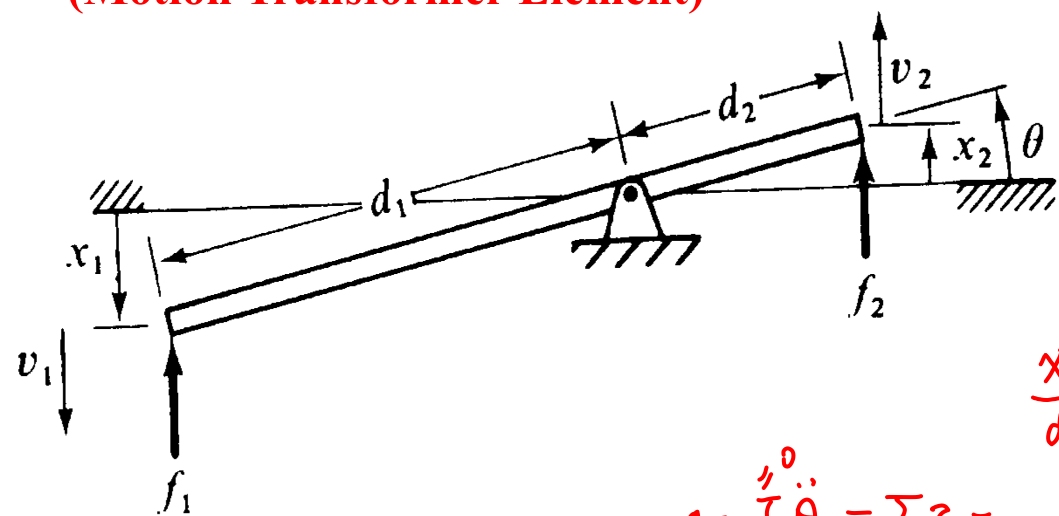
$\Delta PE \Downarrow$ change of potential energy

W: Total work done by the applied torque $\tau(t)$ from time t_0 to t_1

Motion Transfer Elements

- **Lever**

  (Motion Transformer Element)



Assume massless and no joint friction torque

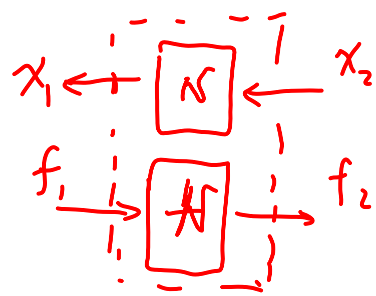
$$\frac{x_1}{d_1} = \sin\theta = \frac{x_2}{d_2} \Rightarrow \frac{x_1}{x_2} = \frac{d_1}{d_2} = N$$

$$0 = \overset{0}{J}\ddot{\theta} = \Sigma\tau = -f_1 d_1 \cos\theta + f_2 d_2 \cos\theta$$

$$\Rightarrow \frac{f_2}{f_1} = \frac{d_1}{d_2} = N$$

Energy Conservation

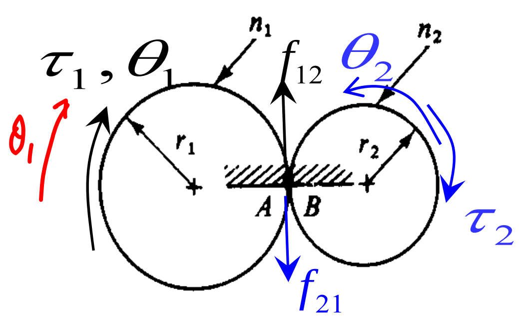
$$f_1 x_1 = f_2 x_2$$

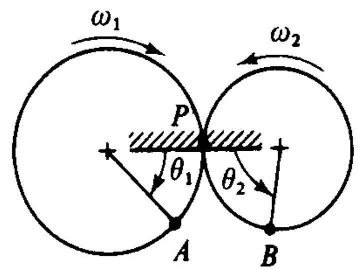


$$f_2 x_2$$

Motion Transfer Elements

- **Ideal Gears**





negligible moment of inertia
and friction torques

Gear 1

$$0 = J_1 \ddot{\theta}_1 = \sum \tau = \tau_1 - f_{12} r_1$$

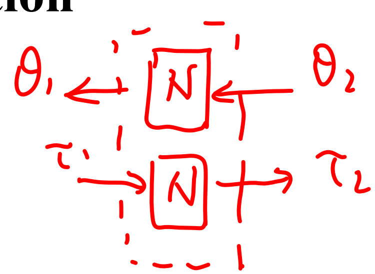
Gear 2

$$0 = J_2 \ddot{\theta}_2 = \sum \tau = -\tau_2 + f_{21} r_2$$

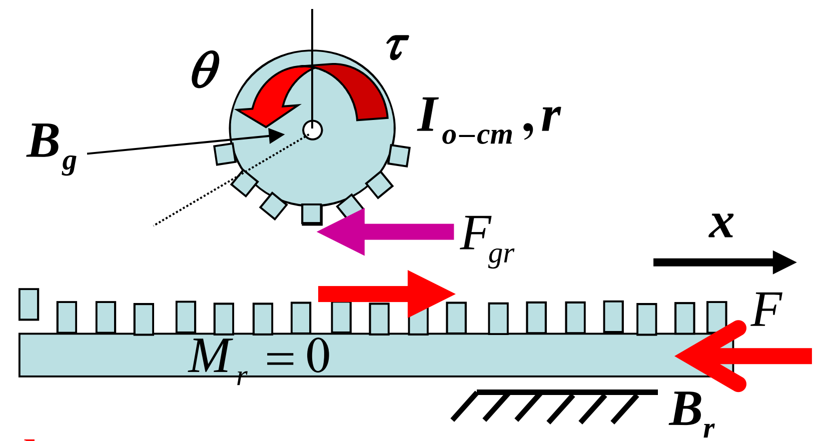
$$\begin{cases} \tau_1 = f_{12} r_1 \\ \tau_2 = f_{21} r_2 \end{cases} \Rightarrow \frac{\tau_1}{\tau_2} = \frac{f_{12} r_1}{f_{21} r_2} = \frac{r_1}{r_2} = \frac{1}{N}$$

$$r_1 \omega_1 = r_2 \omega_2 \Rightarrow \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{1}{N}$$

Energy Conservation



- **Rack and Pinion**



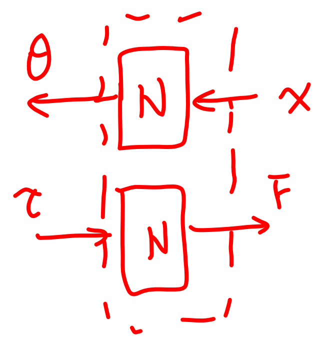
Ideal case (negligible inertias and damping)

$$0 = I_{o-cm} \ddot{\theta} = \tau - F_{gr} r$$

$$0 = M_r \ddot{x} = F_{gr} - F$$

$$\Rightarrow \frac{F}{\tau} = \frac{1}{r}$$

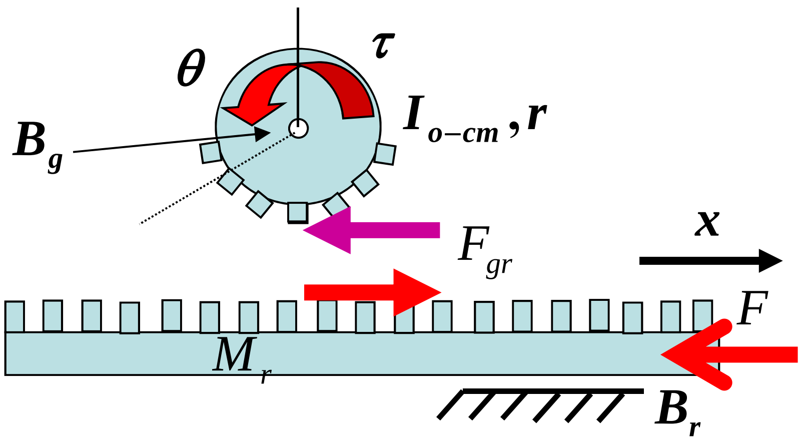
$$\dot{x} = r\dot{\theta} \Rightarrow \frac{\dot{\theta}}{\dot{x}} = \frac{1}{r}$$



$N = \frac{1}{r}$

Example (coupled translation-rotation)

Real world (non-negligible inertias and/or damping)



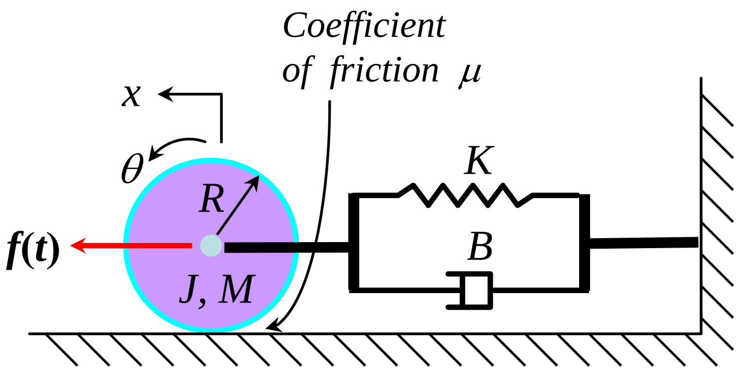
$$\begin{cases} I\ddot{\theta} = \tau - B_g\dot{\theta} - F_{gr}\, r \\ M_r\ddot{x} = F_{gr} - F - B_r\dot{x} \\ x = r\theta \end{cases}$$

$$\Rightarrow \quad \boxed{\underbrace{\left(M_r + \frac{I_{o-cm}}{r^2}\right)}_{M_{eff}}\ddot{x} + \underbrace{\left(B_r + \frac{B_g}{r^2}\right)}_{B_{eff}}\dot{x} = \frac{\tau}{r} - F}$$

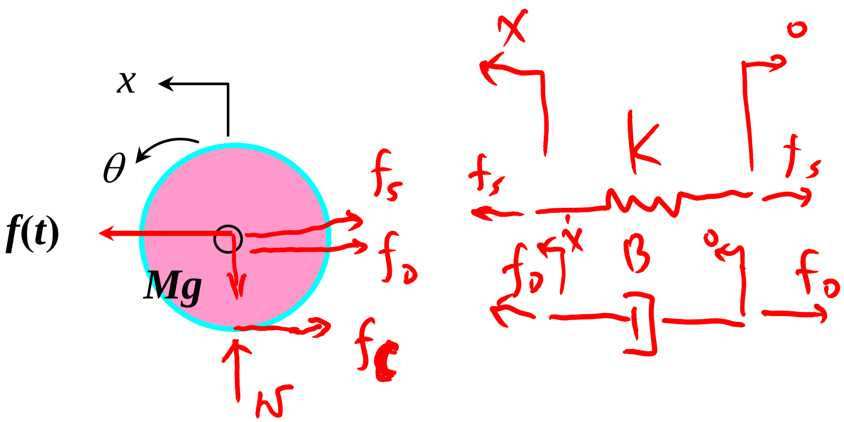
Think about performance of lever and gear train in real life

Example

- Rolling without slipping



FBD:



Elemental Laws:

$$M\ddot{x} = f - f_s - f_D - f_c$$

$$0 = Mg - N$$

$$\frac{1}{2}MR^2\ddot{\theta} = f_c \cdot R$$

$$f_s = K(x + 0)$$

$$f_D = B(\dot{x} - 0)$$

UV: $x, f_s, f_D, f_c, N, \theta$

Roll without slipping:

$$\dot{x} - R\dot{\theta} = 0 \Leftrightarrow x = R\theta$$

$$\Rightarrow f_c = \frac{1}{2}MR\ddot{\theta} = \frac{1}{2}MR\left(\frac{\ddot{x}}{R}\right) = \frac{1}{2}M\ddot{x}$$

Example (cont.)

I/O Model (Input: f, Output: x):

$$M\ddot{x} = f - Kx - B\dot{x} - \frac{1}{2}M\ddot{x}$$

$$\Rightarrow \frac{3}{2}M\ddot{x} + B\dot{x} + Kx = f$$

Q: How would you decide whether or not the disk will slip?

$$|f_c| < \mu_s N = \mu_s M g$$

$$\Leftrightarrow \left|\frac{1}{2}M\ddot{x}\right| < \mu_s M g$$

Q: How will the model be different if the disk rolls and slips ?

$$x \neq R\theta$$

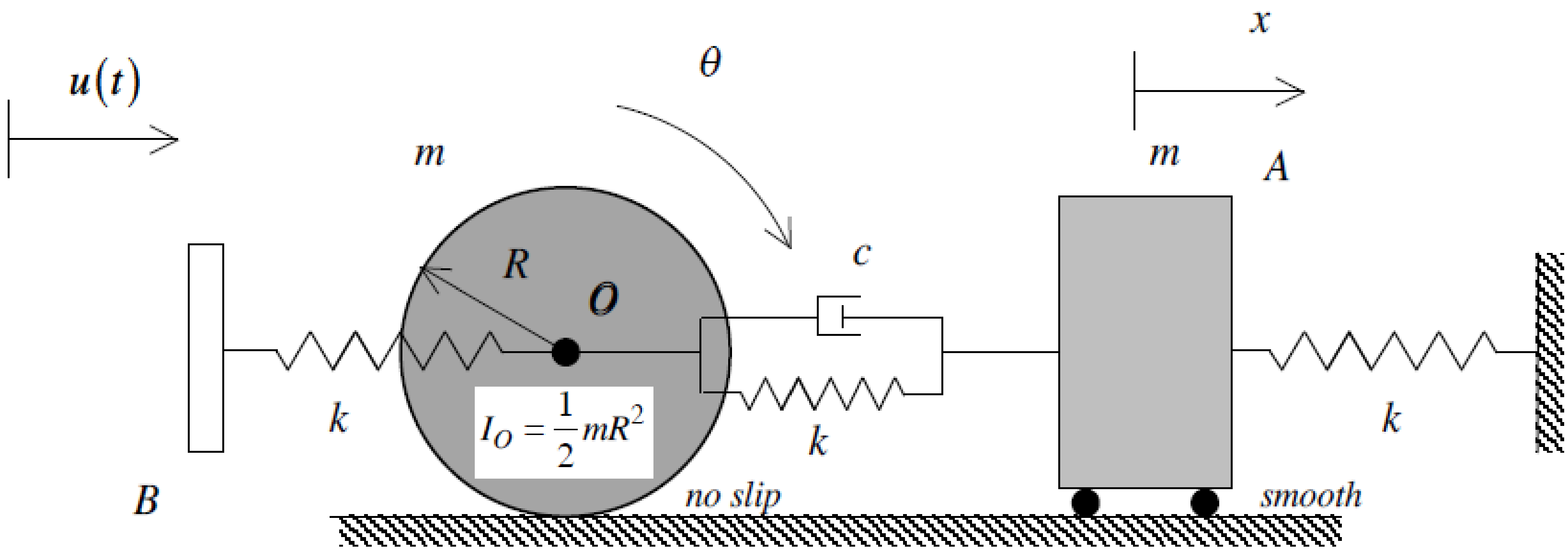
$$f_c = \mu_s M g$$

PURDUE MECHANICAL ENGINEERING

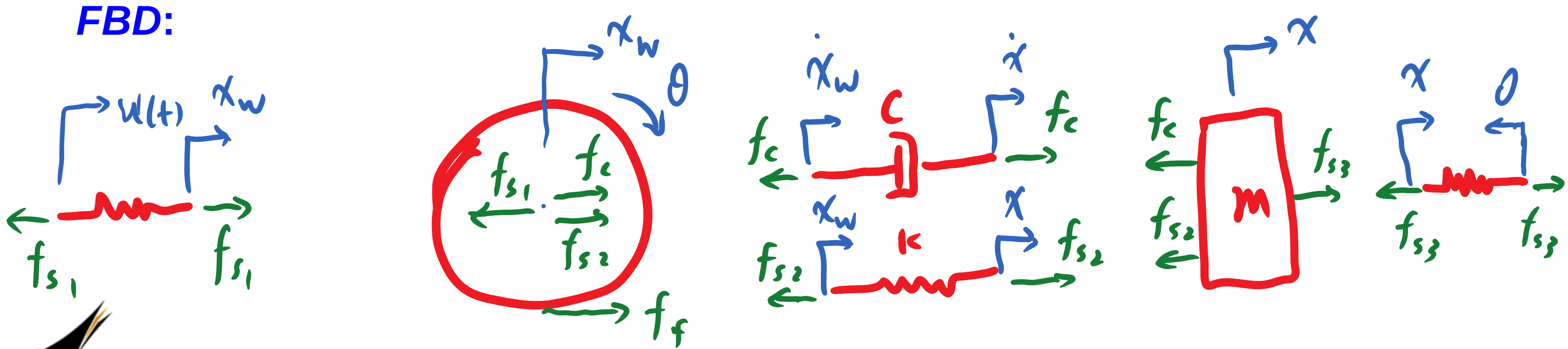
General Mechanical Systems

- **Example**

Derive the differential equation of motions (EOMs) for the system in terms of the outputs x and θ, and the input u(t).



FBD:



Example (cont.)

Elemental Equations:

$f_{s_1} = k(x_w - u)$ ①

$m\ddot{x}_w = -f_{s_1} + f_c + f_{s_2} + f_f$ ②

$I_o\ddot{\theta} = -f_f \cdot R$ ③ $\Rightarrow f_f = -\frac{I_o}{R}\ddot{\theta}$

$f_c = c(\dot{x} - \dot{x}_w)$ ④

$f_{s_2} = k(x - x_w)$ ⑤

$m\ddot{x} = f_{s_3} - f_{s_2} - f_c$ ⑥

$f_{s_3} = k(-x - 0)$ ⑦

no slip: $x_w = R\theta$ ⑧

$m R\ddot{\theta} = -k(\cancel{x_w}\, R\theta - u) + c(\dot{x} - R\dot{\theta}) + k(x - R\theta) + \left(-\frac{I_o}{R}\ddot{\theta}\right)$

$m\ddot{x} = -kx - k(x - R\theta) - c(\dot{x} - R\dot{\theta})$

2 UVs: x & θ

$m\ddot{x} + c\dot{x} - cR\dot{\theta} + 2kx - kR\theta = 0$

$\left(mR + \frac{I_o}{R}\right)\ddot{\theta} - c\dot{x} + cR\dot{\theta} - kx + 2kR\theta = ku$

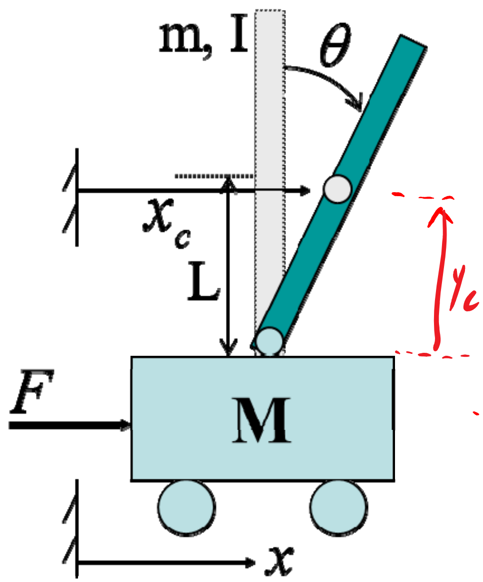
UVs: $f_{s_1}, x_w, f_c, f_{s_2}, f_f, \theta, x, f_{s_3}$

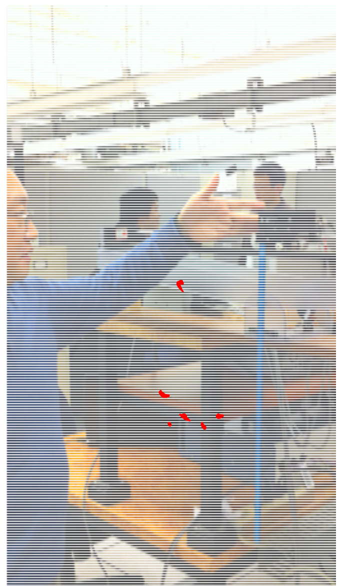
8 UVs = 8 eqs $\Rightarrow$ have all the equations

***Matrix Form*:**

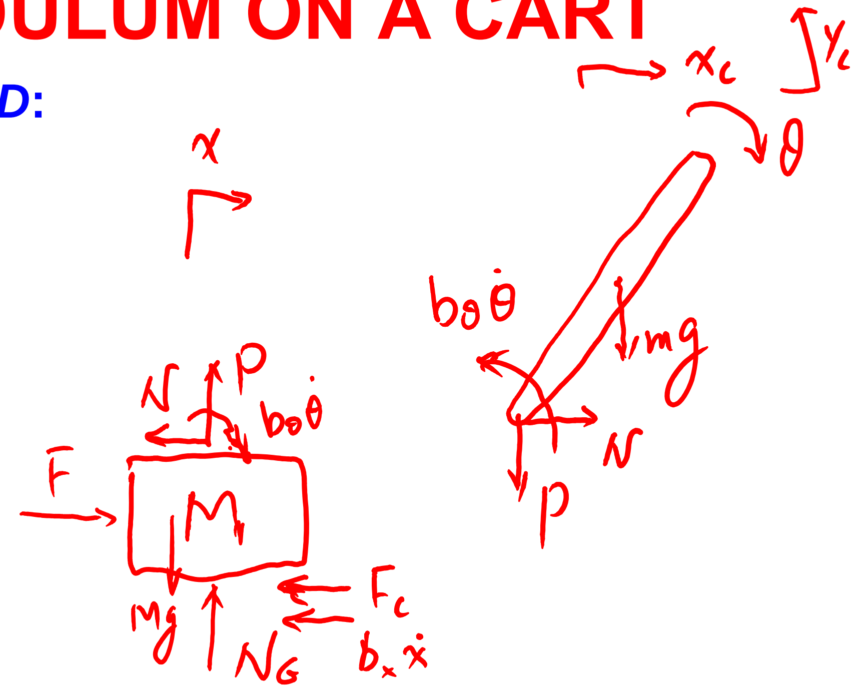
$$\begin{bmatrix} m & 0 \\ 0 & mR + \frac{I_o}{R} \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} +c & -cR \\ -c & cR \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 2k & -kR \\ -k & 2kR \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ k \end{bmatrix} u$$

EXAMPLE - INVETED PENDULUM ON A CART





***FBD*:**



Elemental Equations:

$$M\ddot{x} = F - N - F_c - b_x \dot{x}$$

$$0 = P - Mg + N_G$$

$$m\ddot{x}_c = N$$

$$m\ddot{y}_c = -mg - P$$

$$I\ddot{\theta} = -NL\cos\theta - PL\sin\theta - b_\theta\dot{\theta} \quad (5)$$

UVs:

$x,\ N,\ P,\ N_G,\ x_c,\ y_c,\ \theta$

7 UVs,

Kinematics:

$$x_c = x + L\sin\theta \quad (6)$$

$$y_c = L\cos\theta \quad (7)$$

INVETED PENDULUM ON A CART

- Linearization assuming small angle and $b = 0$

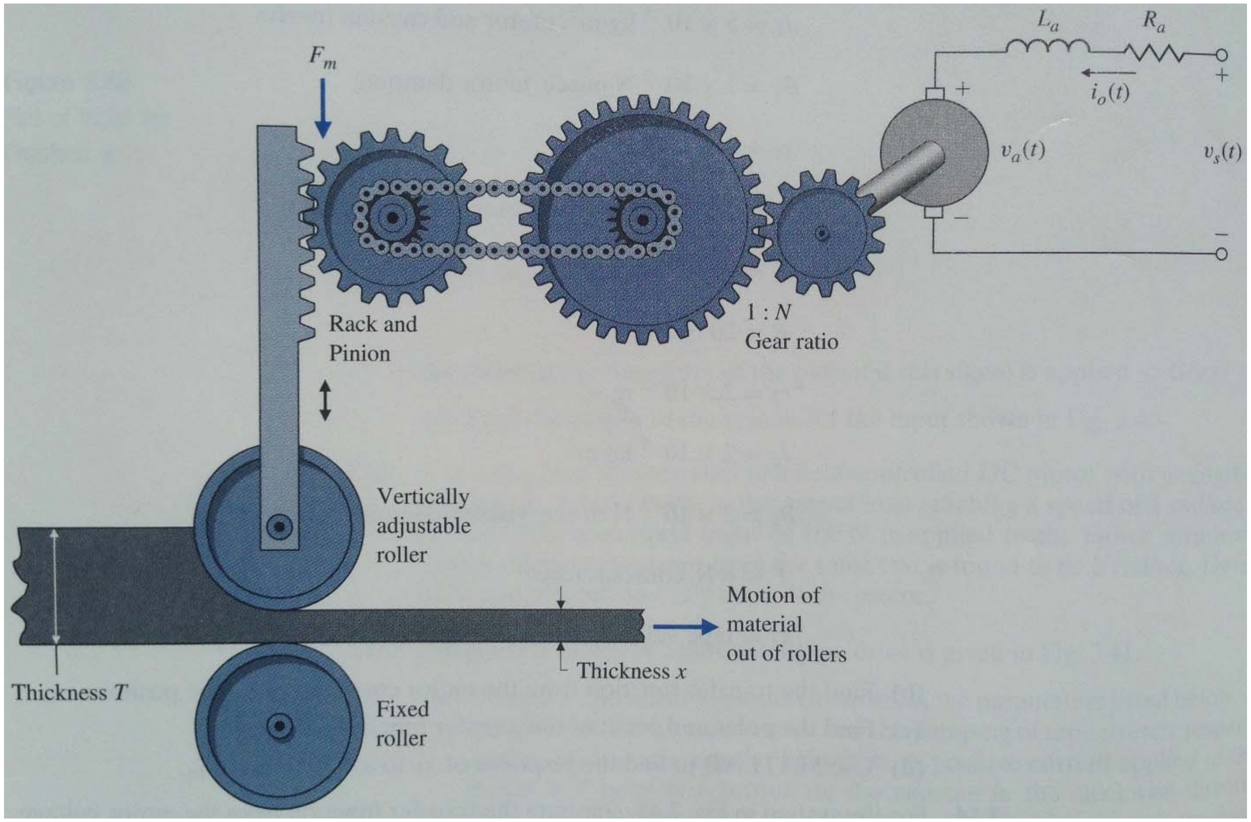
If $\theta \ll 1$, $\cos\theta \simeq 1$, $\sin\theta \simeq \theta$, and neglect all higher-order terms:

$$
\begin{cases}
(M+m)\ddot{x} + mL\ddot{\theta} + b_x\dot{x} = F - F_c \\
mL\ddot{x} + (I + mL^2)\ddot{\theta} + b_\theta\dot{\theta} - mgL\theta = 0
\end{cases}
$$

***Matrix Form*:**

$$
\begin{bmatrix} M+m & mL \\ mL & I+mL^2 \end{bmatrix}
\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}
+ \begin{bmatrix} b_x & 0 \\ 0 & b_\theta \end{bmatrix}
\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}
+ \begin{bmatrix} 0 & 0 \\ 0 & -mgL \end{bmatrix}
\begin{bmatrix} x \\ \theta \end{bmatrix}
= \begin{bmatrix} 1 \\ 0 \end{bmatrix} (F - F_c)
$$

EXAMPLE – CONTINUOUS ROLLING MILL



Suppose that the motion of the adjustable roller has a damping coefficient b, and that the force exerted by the rolled material on the adjustable roller is proportional to the material's change in thickness: $f_s = c(T - x)$. Suppose further that the rack-and-pinion has an effective radius of R and the DC motor can be modeled by a torque input source of τ_m for this problem.

Inputs:

Output:

***FBD*:**

Example (cont.)

Elemental Equations:

***Input-output Form*:**