Closed book. 120 minutes.
Cover page plus five pages of exam.

To receive full credit, show enough work to indicate your logic.

Do not spend time calculating. You will receive full credit if someone with no understanding of probability could simplify your answer to obtain the correct numerical solution.

Recall: The Poisson pmf with mean μ is $f(x) = e^{-\mu}\mu^{x} / x!$ for $x = 0, 1, 2, \ldots$.

Recall: The geometric pmf with probability of success p is $f(x) = p(1-p)^{x-1}$ for $x = 1, 2, 3, \ldots$.
The mean is $1/p$.

Recall: The exponential pdf with rate λ is $f(x) = \lambda e^{-\lambda t}$ for $0 \le t$.
The mean and standard deviation are both $1/\lambda$.

Score ____________________________

Closed book. No calculator.

1. True or False. (If false, write why.) (3 points each)

T *F* (a) The expected value of a nonnegative discrete random variable must be finite.

T *F* (b) In exponential racing, we saw that $P(X \le Y) = \lambda/(\lambda+\mu)$. For this result to be valid, must we have $\lambda < \mu$.

T *F* (c) An irreducible discrete-time Markov chain must have a unique steady-state distribution.

T *F* (d) In any system, the probability of being in an absorbing state is always a nondecreasing function of time.

T *F* (e) The probability of ever returning (even once) to a transient state is zero.

T *F* (f) For all states of a Markov chain to be transient, the number of states must be infinite.

T *F* (g) *Independent increments* refers only to overlapping time intervals.

T *F* (h) A Poisson process has, always, a Poisson number of events during any finite-length interval of time.

T *F* (i) The variance of a sum is the sum of the variances.

T *F* (j) In a $G/G/1$ queueing system, the fraction of time that the server is idle depends only on the mean time between arrivals, $1/\lambda$, and the mean service time, $1/\mu$. In particular, it does not depend upon the form of the distributions G.

2. Consider the queueing system denoted by M/M/2 in Kendall notation. Poisson process arrivals with rate λ. Two servers, each with independent and identically distributed exponential service times. Infinite queue capacity.

(a) (6 points) Model this system by drawing a digraph of the Markov process. Label all nodes and show all rates.

(b) (3 points) T F The model in Part (a) can be interpreted as a birth-and-death process.

(c) (5 points) Does the model of Part (a) explain the procedure for how an arrival to an empty system chooses a server? YES NO

If YES, explain the procedure. If NO, explain why such a procedure is not needed.

(d) (4 points) State condition(s) that must be satisfied for your model in Part (a) to have a steady-state distribution.

3. Consider a continuous-time discrete-state Markov process with states $\{1, 2, ..., m\}$ and homogeneous transition rates λ_{ij}. The associated discrete-time discrete-state Markov chain has the same state space, with a transition being defined by a move from one state to another state, is called the *embedded* Markov chain. Let p_{ij} denote the one-step transition probability from state i to state j for the embedded Markov chain.

(a) (4 points) Write p_{ii} as a function of the λ_{ij}'s.

(b) (4 points) Write p_{ij} as a function of the λ_{ij}'s.

4. (5 points) Consider a NASA computer system for processing data from satellites to ground stations. Data arrives from the satellite deterministically with a rate of 75,000 bits per seconds (i.e., the time between bits is always 1/75000 second). The steady-state average number of bits in the system is 375,000 bits. What is the steady-state average time for a bit to spend in the system?

5. Consider two tasks. T_i, the time of the ith task, is exponential with rate λ_i, for $i = 1, 2$.

Consider two systems. In System A, both tasks must be performed sequentially. In System B, a worker must perform one task; let p_i denote the probability that the ith task is to be performed. Let T_A denote the total time for System A; let T_B denote the total time for System B.

(a) (4 points) Write T_A in terms of the other notation. (You don't need to use all of the various notations.)

(b) (4 points) T F $\mathrm{E}(T_B) \le \mathrm{E}(T_A)$.

(c) (4 points) T F $\mathrm{Var}(T_B) \le \mathrm{Var}(T_A)$.

(d) (4 points) T F The PASTA (Poisson Arrivals See Time Averages) property guarantees that the average times (for both systems) is Poisson.

(e) (4 points) T F The independent-increments property holds; that is, T_1 and T_2 are independent events.

6. (4 points) Consider the times that car accidents occur in the city of West Lafayette. Comment on the appropriateness of modeling these times with a Poisson process with constant rate λ. (Consider all assumptions of the Poisson process, not just the constant rate.)

7. Consider a Markov Reward Process with one-step transition matrix $P = (p_{ij})$, reward matrix $R = (r_{ij})$ and discount factor α.

(a) (5 points) In words, explain the meaning of $\sum_{j=1}^{\infty} p_{ij}\, r_{ij}$.

(b) (4 points) T F Although each reward r_{ij} can be any real number, the expected reward must be positive (by definition of "reward").

(c) For each statement about α, agree or disagree. If you disagree, explain why.

(i) (2 points) If the rewards are money, then economics suggest that a payment in the future is worth less than a payment now. Therefore, $\alpha < 1$ is a common assumption. AGREE DISAGREE

(ii) (2 points) $\alpha \le 1$. Otherwise, expected rewards (beyond the immediate rewards) are undefined (because the definition involves division by zero or the inverse of a less-than-full-rank matrix). AGREE DISAGREE

(iii) (2 points) $\alpha < 1$. Otherwise, steady-state probabilities do not exist.

AGREE DISAGREE