ECE 440 – Spring 2019
Stationary and Ergodicity

Summary

A random process $X(t)$ can be understood as a sequence (possibly continuous) of scalar random variables indexed by time. According to this definition, you have a random variable for each value of t, taking scalar values. $X(t_1), X(t_2), \ldots, X(t_N)$ are just random variables which might be correlated...or not.

A random process $X(t)$ can also be understood as a random variable which instead of taking scalar values, takes **function** values. Imagine a die with a large (or infinite) number of faces, and a function $x_i(t)$ drawn on the i-th face. An instance of the random process consists of throwing the die and getting a function (for example $x_i(t)$).

I will use capital letters to represent random processes (e.g., $X(t)$) or random variables (e.g., $X(3)$). I will use lower case letters to represent deterministic values or functions (e.g., $x_1(t)$). The joint probability density function for random variables $X(t_1), X(t_2), \ldots$ will be denoted by $f_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N)$. Observe that the random variables are capitalized, but their deterministic values $x_1, x_2, \ldots$ are not.

The **expected value** of a random variable X is defined as $E[X] = \int_{-\infty}^{\infty} \alpha f_X(\alpha) d\alpha$ and often denoted μ_X. The distribution $f_X(\alpha)$ will often be restricted to a specific domain, which will set the limits of the integration. The expected value of a random process $X(t)$ is defined as $E[X(t)] = \int_{-\infty}^{\infty} \alpha f_{X(t)}(\alpha) d\alpha$. It will, in general, depend on time and would be denoted as $\mu_X(t)$. However, see the effect of stationarity below.

The **variance** of a random variable X is defined as $\sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{\infty} (\alpha - \mu_X)^2 f_X(\alpha) d\alpha$. If the variable has zero mean, then the variance can be found as $E[X^2]$.

The **covariance** of two random variables X, Y is defined as $\mu_{XY} = E[(X - \mu_x)(Y - \mu_y)]$. After normalizing by $\sigma_x \sigma_y$, it specifies how correlated they are.

The autocorrelation of a random process $X(t)$ at t_1 and t_2 is defined as $E[X(t_1)X(t_2)]$.

The **time average** of a deterministic function $x(t)$ is defined as $< x(t) >= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$.

- A random process $X(t)$ is **strict sense stationary** if the joint distribution for random variables $X(t_1), X(t_2), \ldots, X(t_N)$ is the same as that for $X(t_1 + \Delta), X(t_2 + \Delta), \ldots, X(t_N + \Delta)$, for any value of $t_1, t_2, \ldots, t_N$ and for any value of Δ.

  In particular, if $X(t)$ is strict sense stationary, then $f_{X(t_1)}(\alpha) = f_{X(t_1+\Delta)}(\alpha)$ for any Δ. This implies that the distribution of the variable $X(t_1)$ is the same as that of the variable $X(t_2)$ for any t_1 and t_2. Hence, the distribution is the same for every time instance. If the distribution is the same, then the mean, the variance, covariance and any other moment you can think of are also the same.

  In particular, the mean and the variance of a strict sense stationary process $X(t)$ are independent of time. All their time-indexed random variables have the same mean μ_X and variance σ_X^2. If we compute the expected value of the process $E[X(t)]$, it will not depend on time, and can be denoted as μ_X. A similar argument can be followed for variance. Furthermore, the covariance of $X(t_1)$ and $X(t_2)$ does not depend on the specific time instants t_1 and t_2, but only on how far apart they are $\tau = t_2 - t_1$. The **autocorrelation** of $X(t)$ at t_1 and t_2 also does not depend on the time instants but only on the time difference. It is denoted $R_X(\tau)$.

  In this class, when we say that a process is stationary, you may assume that we mean wide sense stationary.

- A random process $X(t)$ is **wide sense stationary** (WSS) if its mean and variance are time-independent and its autocorrelation only depends on the time difference. This can be understood as follows: if you take any two random variables from this process $X(t_1)$ and $X(t_2)$, they both have the same mean and variance, and their covariance only depends on the time difference τ.

  It is clear that strict sense stationarity implies wide sense stationarity, but not the other way. However, if the process is Gaussian (i.e., any subset of variables from the process follows a jointly Gaussian distribution), then strict sense is the same as wide sense.

- A stationary random process $X(t)$ is **ergodic** if all the time averages over any specific instance of the process $x_i(t)$ give the same value as the expectations over the possible instances. That is, knowing one instance of the process gives us all the statistical information about the whole process. Going back to the die example, we throw the die once, get a function back, and have all we need to characterize the whole process. Time averages over that function will behave in the same way as expectations over the instances.

  In particular, we can find the mean and variance of $X(t_1)$ for any t_1 by finding the time average $\mu_X =< x_i(t) >$ and $< (x_i(t) - \mu_X)^2 >$, respectively. Similarly, we can find how correlated $X(t_1)$ and $X(t_1 + \tau)$ are by looking at how correlated $x_i(t)$ is with $x_i(t + \tau)$. Specifically, $R_X(\tau) =< x_1(t)x_1(t + \tau) >$. In other words, the autocorrelation of $X(t)$ as a process is the same as the autocorrelation of any specific instance $x_1(t)$ as a deterministic function.

  If the process is not stationary, then it cannot be ergodic.

Checking whether a given random process is strict sense stationary or ergodic would require making sure that all the conditions in the definitions above hold, but this is often tedious in practice. In the exams, I will at most ask you:

- Check whether a process is WSS: Answer YES if all the following are true: 1) the mean is time-independent, 2) the variance is time-independent, 3) the autocorrelation only depends on time differences.

- Check whether a process is Ergodic: Answer YES if all the following are true: 1) the process is WSS, 2) $E[X(t)] =< x_i(t) >$ for any instance i, 3) $E[(X(t) - \mu_X)^2] =< (x_i(t) - \mu_X)^2 >$ for any instance i, 4) $R_X(\tau) =< x_i(t)x_i(t + \tau) >$ for any instance i.