Markov Random Fields and Stochastic Image Models

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   (d) Parameter estimation
4. Application of MRF's to Segmentation
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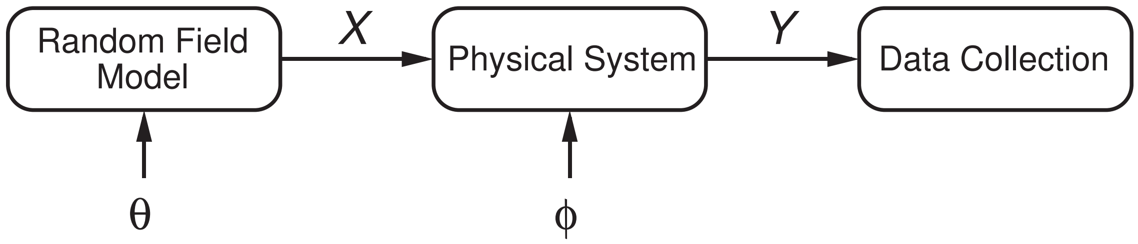
The Bayesian Approach

θ - Random field model parameters

X - Unknown image

ϕ - Physical system model parameters

Y - Observed data

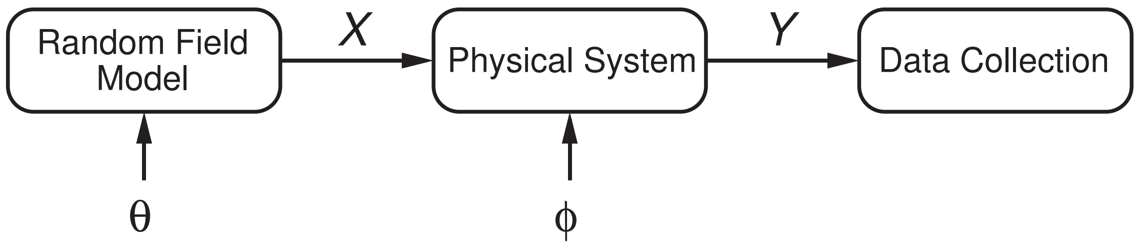


- Random field may model:
  - Achromatic/color/multispectral image
  - Image of discrete pixel classifications
  - Model of object cross-section
- Physical system may model:
  - Optics of image scanner
  - Spectral reflectivity of ground covers (remote sensing)
  - Tomographic data collection

Bayesian Versus Frequentist?

- How does the Bayesian approach differ?
  - Bayesian makes assumptions about prior behavior.
  - Bayesian requires that you choose a model.
  - A **good** prior model can improve accuracy.
  - But model mismatch can impair accuracy
- When should you use the frequentist approach?
  - When (# of data samples)>>(# of unknowns).
  - When an accurate prior model does not exist.
  - When prior model is not needed.
- When should you use the Bayesian approach?
  - When (# of data samples)≈(# of unknowns).
  - When model mismatch is tolerable.
  - When accuracy without prior is poor.

Examples of Bayesian Versus Frequentist?

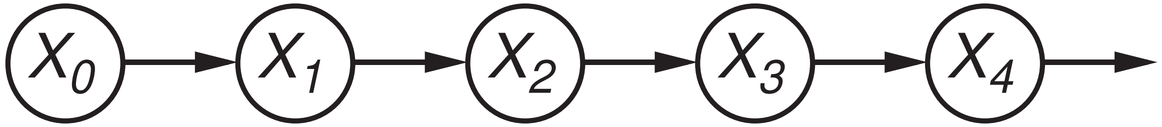


- Bayesian model of image X
  - (# of image points)≈(# of data points.)
  - Images have unique behaviors which may be modeled.
  - Maximum likelihood estimation works poorly.
  - Reduce model mismatch by estimating parameter θ.
- Frequentist model for θ and ϕ
  - (# of model parameters)<<(# of data points.)
  - Parameters are difficult to model.
  - Maximum likelihood estimation works well.

Markov Chains

- Topics to be covered:
  - 1-D properties
  - Parameter estimation
  - 2-D Markov Chains
- **Notation:** Upper case $\Rightarrow$ Random variable

Markov Chains



- Definition of (homogeneous) Markov chains

$$p(x_n|x_i \; i < n) = p(x_n|x_{n-1})$$

- Therefore, we may show that the probability of a sequence is given by

$$p(x) = p(x_0) \prod_{n=1}^{N} p(x_n|x_{n-1})$$

- Notice: X_n is **not** independent of X_{n+1}

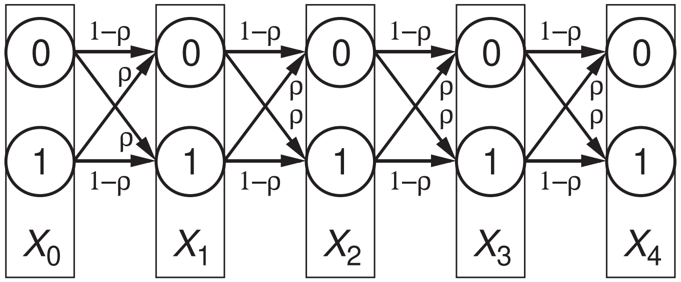
$$p(x_n|x_i \; i \neq n) = p(x_n|x_{n-1}, x_{n+1})$$

Parameters of Markov Chain

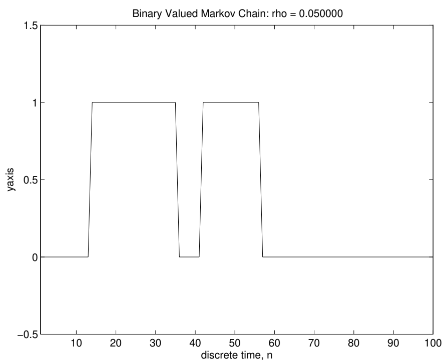
- Transition parameters are:

$$\theta_{j,i} = p(x_n = i | x_{n-1} = j)$$

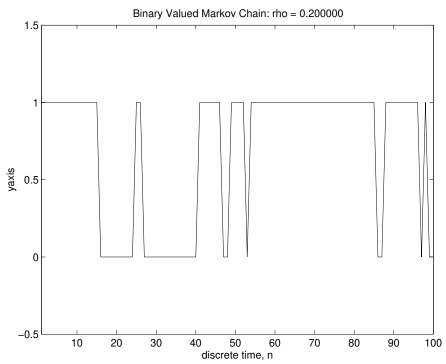
- Example: $\theta = \begin{bmatrix} 1-\rho & \rho \\ \rho & 1-\rho \end{bmatrix}$



- ρ is the probability of changing state.



$\rho = 0.05$



$\rho = 0.2$

Parameter Estimation for Markov Chains

- Maximum likelihood (ML) parameter estimation

$$\hat{\theta} = \arg\max_{\theta} p(x|\theta)$$

- For Markov chain

$$\hat{\theta}_{j,i} = \frac{h_{j,i}}{\sum_k h_{j,k}}$$

where $h_{j,i}$ is the histogram of transitions

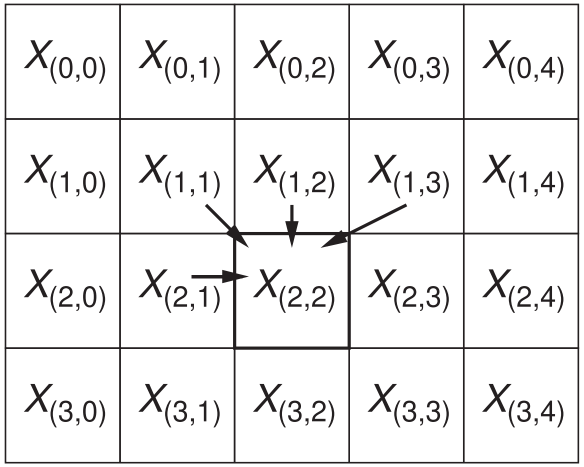
$$h_{j,i} = \sum_n \delta(x_n = i \,\&\, x_{n-1} = j)$$

- Example

$$x_n = 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1$$

$$\theta = \begin{bmatrix} h_{0,0} & h_{0,1} \\ h_{1,0} & h_{1,1} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 6 \end{bmatrix}$$

2-D Markov Chains



- Advantages:
  - Simple expressions for probability
  - Simple parameter estimation
- Disadvantages:
  - No natural ordering of pixels in image
  - Anisotropic model behavior

Discrete State Markov Random Fields

- Topics to be covered:
  - Definitions and theorems
  - 1-D MRF's
  - Ising model
  - M-Level model
  - Line process model

Markov Random Fields

- Noncausal model
- Advantages of MRF's
  - Isotropic behavior
  - Only local dependencies
- Disadvantages of MRF's
  - Computing probability is difficult
  - Parameter estimation is difficult
- Key theoretical result: Hammersley-Clifford theorem

Definition of Neighborhood System and Clique

- Define

  S - set of lattice points

  s - a lattice point, $s \in S$

  X_s - the value of X at s

  ∂s - the neighboring points of s

- A neighborhood system ∂s must be symmetric

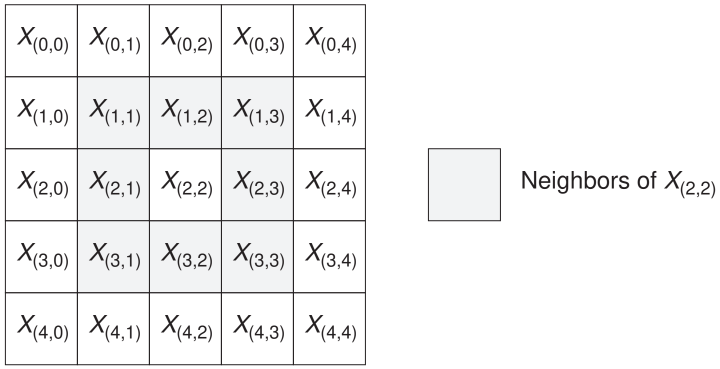
$$r \in \partial s \Rightarrow s \in \partial r \ \text{ also } s \notin \partial s$$

- A clique is a set of points, c, which are all neighbors of each other

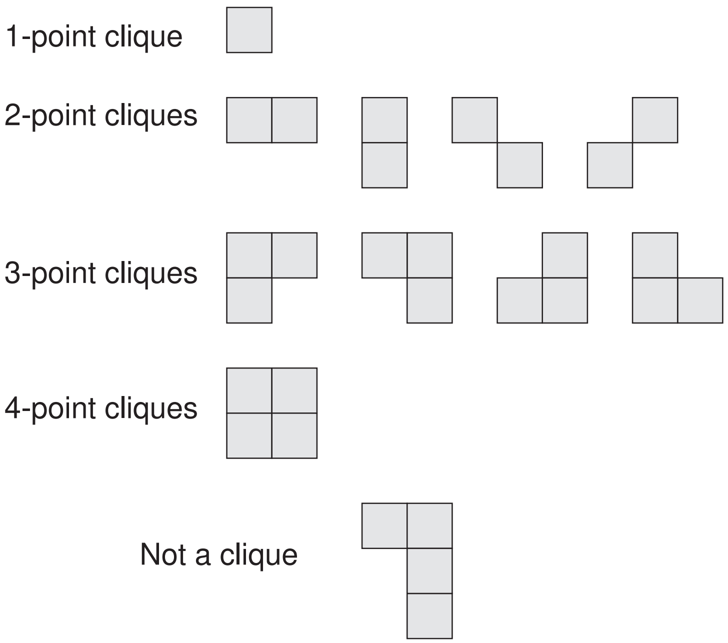
$$\forall s, r \in c, \; r \in \partial s$$

Example of Neighborhood System and Clique

- Example of 8 point neighborhood



- Example of cliques for 8 point neighborhood



Gibbs Distribution

x_c - The value of X at the points in clique c.

$V_c(x_c)$ - A potential function is any function of x_c.

- A (discrete) density is a Gibbs distribution if

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{c \in \mathcal{C}} V_c(x_c) \right\}$$

  $\mathcal{C}$ is the set of all cliques

  Z is the normalizing constant for the density.

- Z is known as the **partition function**.
- $U(x) = \sum_{c \in \mathcal{C}} V_c(x_c)$ is known as the **energy function**.

Markov Random Field

- Definition: A random object X on the lattice S with neighborhood system ∂s is said to be a Markov random field if for all $s \in S$

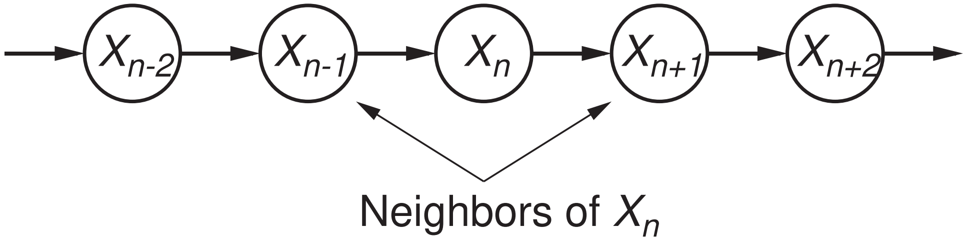
$$p(x_s | x_r \text{ for } r \neq s) = p(x_s | x_{\partial r})$$

Hammersley-Clifford Theorem[14]

$$\left(\begin{array}{c} X \text{ is a Markov random field} \\ \& \\ \forall x,\ P\{X = x\} > 0 \end{array} \right) \Longleftrightarrow \left(\begin{array}{c} P\{X = x\} \text{ has the form} \\ \text{of a Gibbs distribution} \end{array} \right)$$

- Gives you a method for writing the density for a MRF
- Does not give the value of Z, the partition function.
- Positivity, $P\{X = x\} > 0$, is a technical condition which we will generally assume.

Markov Chains are MRF's



- Neighbors of n are $\partial n = \{n-1, n+1\}$
- Cliques have the form $c = \{n-1, n\}$
- Density has the form

$$
\begin{aligned}
p(x) &= p(x_0) \prod_{n=1}^{N} p(x_n | x_{n-1}) \\
&= p(x_0) \exp\left\{ \sum_{n=1}^{N} \log p(x_n | x_{n-1}) \right\}
\end{aligned}
$$

- The potential functions have the form

$$
V(x_n, x_{n-1}) = \log p(x_n | x_{n-1})
$$

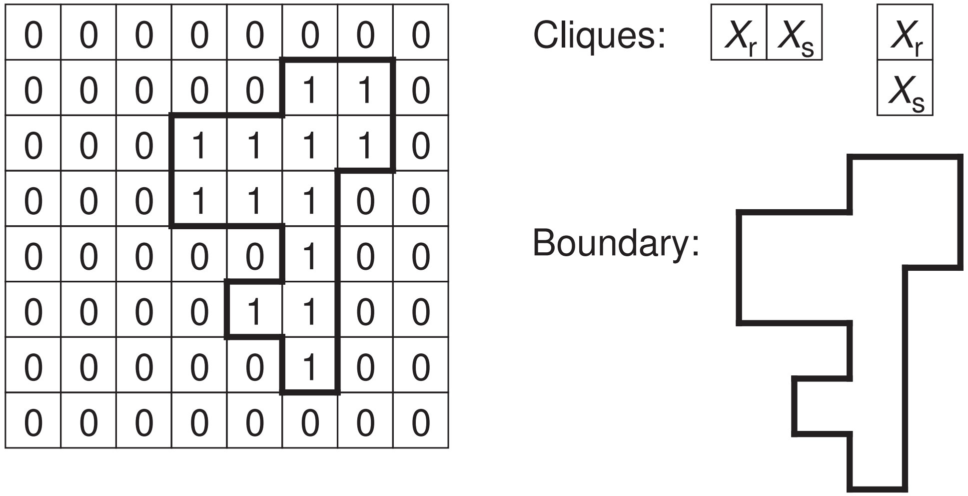
1-D MRF's are Markov Chains

- Let X_n be a 1-D MRF with $\partial n = \{n-1, n+1\}$
- The discrete density has the form of a Gibbs distribution

$$p(x) = p(x_0) \exp\left\{ \sum_{n=1}^{N} V(x_n, x_{n-1}) \right\}$$

- It may be shown that this is a Markov Chain.
- Transition probabilities may be difficult to compute.

The Ising Model: A 2-D MRF[100]



- Potential functions are given by

$$V(x_r, x_s) = \beta\delta(x_r \neq x_s)$$

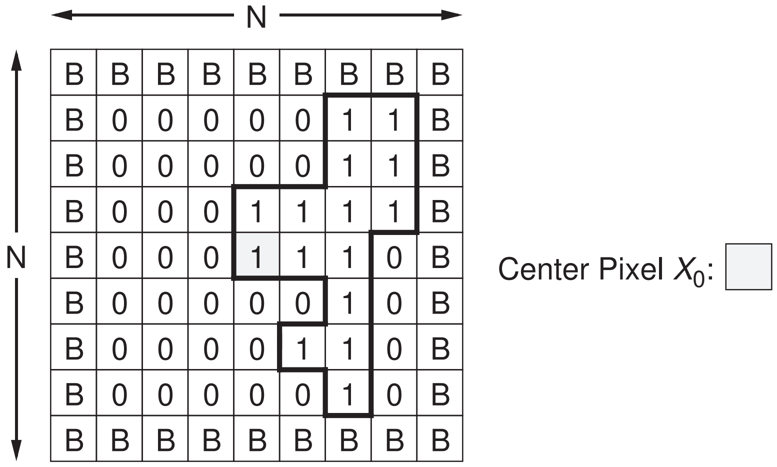
where β is a model parameter.

- Energy function is given by

$$\sum_{c\in\mathcal{C}} V_c(x_c) = \beta(\text{Boundary length})$$

- Longer boundaries $\Rightarrow$ less probable

Critical Temperature Behavior[127, 126, 100]



- $\frac{1}{\beta}$ is analogous to temperature.
- Peierls showed that for $\beta > \beta_c$

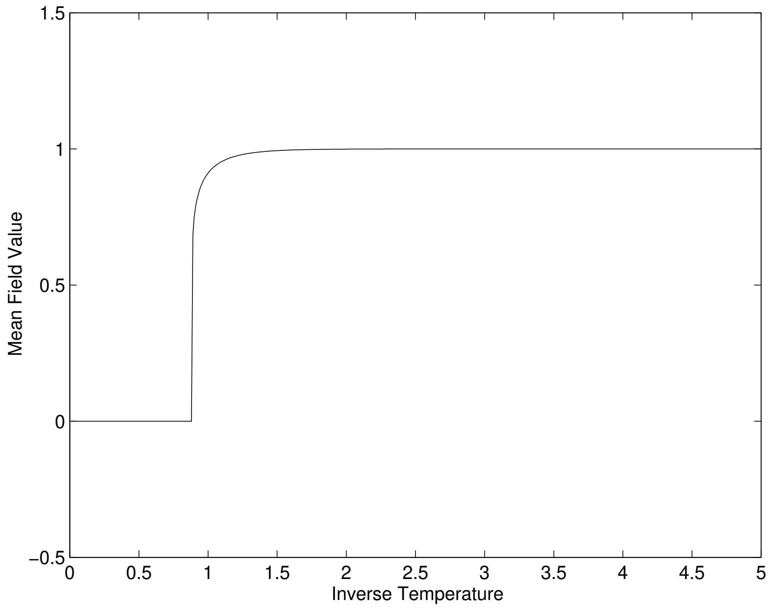
$$\lim_{N\to\infty} P(X_0 = 0|B = 0) \neq \lim_{N\to\infty} P(X_0 = 0|B = 1)$$

- The effect of the boundary does not diminish as $N \to \infty$!
- $\beta_c \approx .88$ is known as the critical temperature.

Critical Temperature Analysis[122]

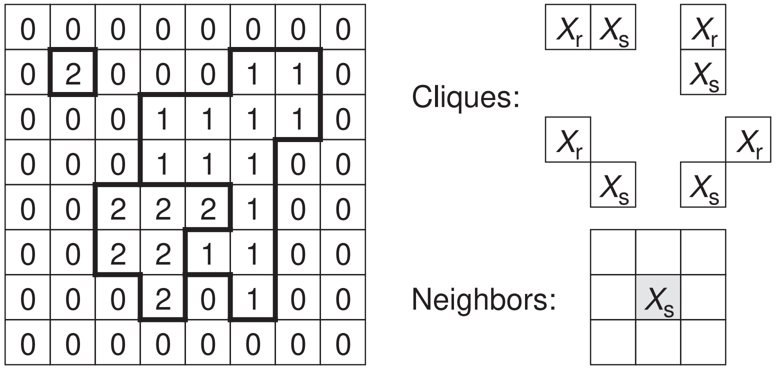
- Amazingly, Onsager was able to compute

$$E[X_0|B=1] = \begin{cases} \left(1 - \frac{1}{(\sinh(\beta))^4}\right)^{1/8} & \text{if } \beta > \beta_c \\ 0 & \text{if } \beta < \beta_c \end{cases}$$



- Onsager also computed an analytic expression for $Z(T)$!

M-Level MRF[16]



- Define $\mathcal{C}_1 \triangleq$ (hor./vert. cliques) and $\mathcal{C}_2 \triangleq$ (diag. cliques)
- Then

$$V(x_r, x_s) = \begin{cases} \beta_1 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_1 \\ \beta_2 \delta(x_r \neq x_s) & \text{for } \{x_r, x_s\} \in \mathcal{C}_2 \end{cases}$$

- Define

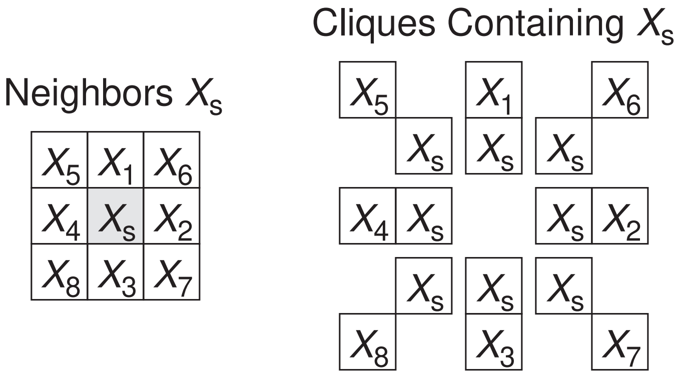
$$t_1(x) \triangleq \sum_{\{s,r\}\in\mathcal{C}_1} \delta(x_r \neq x_s)$$

$$t_2(x) \triangleq \sum_{\{s,r\}\in\mathcal{C}_2} \delta(x_r \neq x_s)$$

- Then the probability is given by

$$p(x) = \frac{1}{Z} \exp\left\{-(\beta_1 t_1(x) + \beta_2 t_2(x))\right\}$$

Conditional Probability of a Pixel



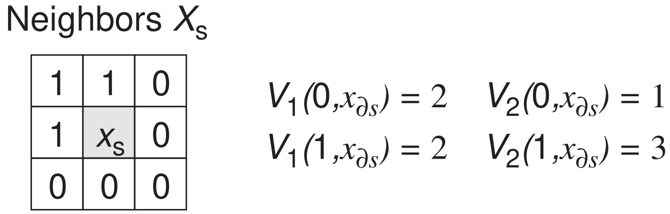
- The probability of a pixel given all other pixels is

$$p(x_s|x_{i\neq s}) = \frac{\frac{1}{Z}\exp\left\{-\sum_{c\in\mathcal{C}} V_c(x_c)\right\}}{\sum_{x_s=0}^{M-1}\frac{1}{Z}\exp\left\{-\sum_{c\in\mathcal{C}} V_c(x_c)\right\}}$$

- Notice: Any term $V_c(x_c)$ which does not include x_s cancels.

$$p(x_s|x_{i\neq s}) = \frac{\exp\left\{-\beta_1\sum_{i=1}^{4}\delta(x_s\neq x_i) - \beta_2\sum_{i=5}^{8}\delta(x_s\neq x_i)\right\}}{\sum_{x_s=0}^{M-1}\exp\left\{-\beta_1\sum_{i=1}^{4}\delta(x_s\neq x_i) - \beta_2\sum_{i=5}^{8}\delta(x_s\neq x_i)\right\}}$$

Conditional Probability of a Pixel (Continued)



- Define

$$\begin{aligned} v_1(x_s, \partial x_s) &\triangleq \text{\# of horz./vert. neighbors} \neq x_s \\ v_2(x_s, \partial x_s) &\triangleq \text{\# of diag. neighbors} \neq x_s \end{aligned}$$

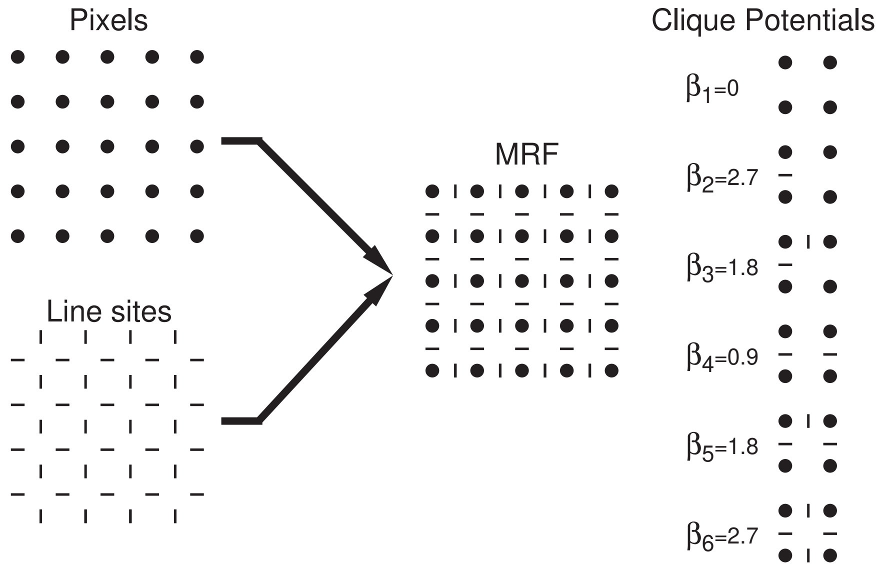
- Then

$$p(x_s|x_{i\neq s}) = \frac{1}{Z'} \exp\left\{-\beta_1 v_1(x_s, \partial x_s) - \beta_2 v_2(x_s, \partial x_s)\right\}$$

where Z' is an easily computed normalizing constant

- When $\beta_1, \beta_2 > 0$, X_s is most likely to be the majority neighboring class.

Line Process MRF [68]



- Line sites fall between pixels
- The values $\beta_1, \cdots, \beta_2$ determine the potential of line sites
- The potential of pixel values is

$$V(x_s, x_r, l_{r,s}) = \begin{cases} (x_s - x_r)^2 & \text{if } l_{r,s} = 0 \\ 0 & \text{if } l_{r,s} = 1 \end{cases}$$

- The field is
  - Smooth between line sites
  - Discontinuous at line sites

Simulation

- Topics to be covered:
  - Metropolis sampler
  - Gibbs sampler
  - Generalized Metropolis sampler

Generating Samples from a Gibbs Distribution

- How do we generate a random variable X with a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\{-U(x)\}$$

- Generally, this problem is difficult.
- Markov Chains can be generated sequentially
- Non-causal structure of MRF's makes simulation difficult.

The Metropolis Sampler[118, 100]

- How do we generate a sample from a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\{-U(x)\}$$

- Start with the sample x^k, and generate a new sample W with probability $q(w|x^k)$.

  Note: $q(w|x^k)$ must be symmetric.

$$q(w|x^k) = q(x^k|w)$$

- Compute $\Delta E(W) = U(W) - U(x^k)$, then do the following:

  If $\Delta E(W) < 0$

  – Accept: $X^{k+1} = W$

  If $\Delta E(W) \geq 0$

  – Accept: $X^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
  – Reject: $X^{k+1} = x^k$ with probability $1 - \exp\{-\Delta E(W)\}$

Ergodic Behavior of Metropolis Sampler

- The sequence of random fields, X^k, form a Markov chain.
- Let $p(x^{k+1}|x^k)$ be the transition probabilities of the Markov chain.
- Then X^k is reversible

$$p(x^{k+1}|x^k)\exp\{-U(x^k)\} = \exp\{-U(x^{k+1})\}p(x^k|x^{k+1})$$

- Therefore, if the Markov chain is irreducible, then

$$\lim_{k\to\infty} P\{X^k = x\} = \frac{1}{Z}\exp\{-U(x)\}$$

- If every state can be reached, then as $k \to \infty$, X^k will be a sample from the Gibbs distribution.

Example Metropolis Sampler for Ising Model

	0	
1	x_s	0
	0	

- Assume $x_s^k = 0$.
- Generate a binary R.V., W, such that $P\{W = 0\} = 0.5$.

$$\begin{aligned} \Delta E(W) &= U(W) - U(x_s^k) \\ &= \begin{cases} 0 & \text{if } W = 0 \\ 2\beta & \text{if } W = 1 \end{cases} \end{aligned}$$

If $\Delta E(W) < 0$

- Accept $X_s^{k+1} = W$

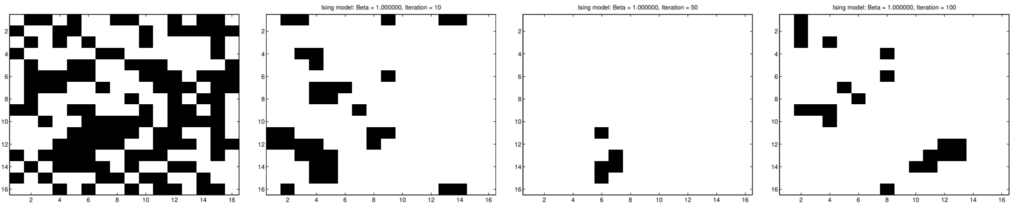
If $\Delta E(W) \geq 0$

- Accept: $X_s^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
- Reject: $X_s^{k+1} = x_s^k$ with probability $1 - \exp\{-\Delta E(W)\}$

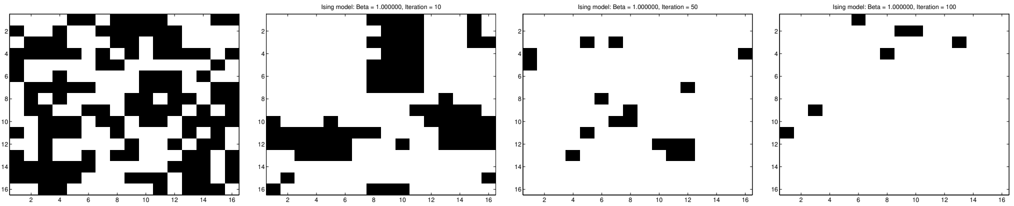
- Repeat this procedure for each pixel.
- **Warning:** for $\beta > \beta_c$ convergence can be extremely slow!

Example Simulation for Ising Model($\beta = 1.0$)

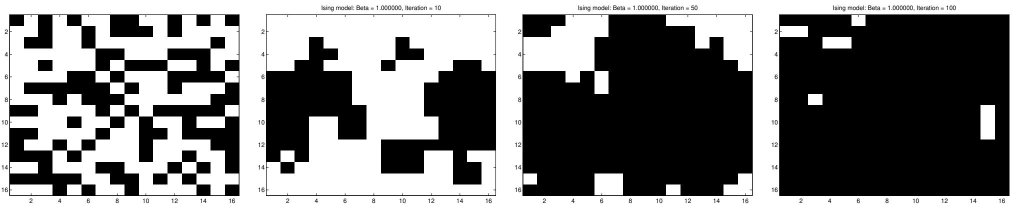
- Test 1



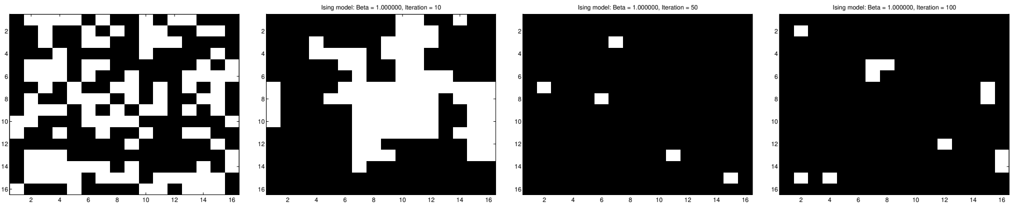
- Test 2



- Test 3



- Test 3



Advantages and Disadvantages of Metropolis Sampler

- Advantages
  - Can be implemented whenever ΔE is easy to compute.
  - Has guaranteed geometric convergence.
- Disadvantages
  - Can be slow if there are many rejections.
  - Is constrained to use a symmetric transition function $q(x^{k+1}|x^k)$.

Gibbs Sampler[68]

- Replace each point with a sample from its conditional distribution

$$p(x_s|x_i^k \; i \neq s) = p(x_s|x_{\partial s})$$

- Scan through all the points in the image.
- Advantage
  - Eliminates need for rejections $\Rightarrow$ faster convergence
- Disadvantage
  - Generating samples from $p(x_s|x_{\partial s})$ can be difficult.

Generalized Metropolis Sampler[80, 129]

- Hastings and Peskun generalized the Metropolis sampler for transition functions $q(w|x^k)$ which are not symmetric.
- The acceptance probability is then

$$\alpha(x_s^k, w) = \min\left\{1, \frac{q(x^k|w)}{q(w|x^k)} \exp\{-\Delta E(w)\}\right\}$$

- Special cases

$$q(w|x^k) = q(x^k|z) \Rightarrow \text{conventional Metropolis}$$
$$q(w_s|x^k) = p(x_s^k|x_{\partial s}^k)\big|_{x_s^k=w_s} \Rightarrow \text{Gibbs sampler}$$

- Advantage
  - Transition function may be chosen to minimize rejections[76]

Parameter Estimation for Discrete State MRF's

- Topics to be covered:
  - Why is it difficult?
  - Coding/maximum pseudolikehood
  - Least squares

Why is Parameter Estimation Difficult?

- Consider the ML estimate of β for an Ising model.
- Remember that

$$t_1(x) = (\#\text{ horz. and vert. neighbors of different value.})$$

- Then the ML estimate of β is

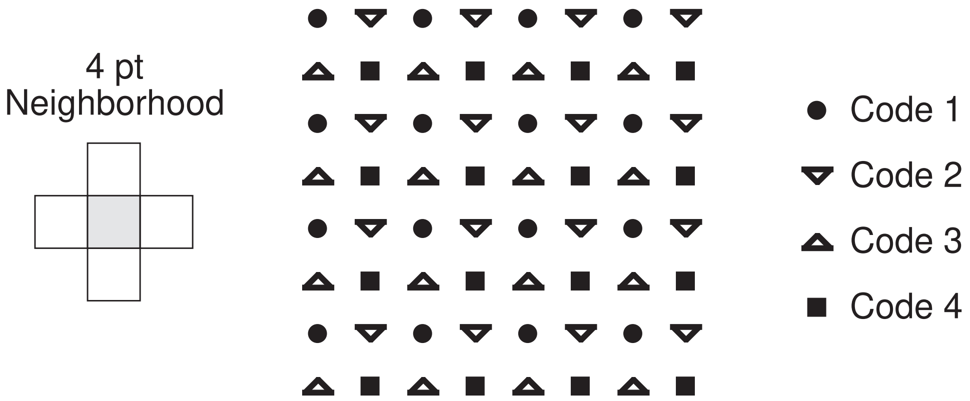
$$\begin{aligned}\hat{\beta} &= \arg\max_{\beta} \left\{ \frac{1}{Z(\beta)} \exp\left\{-\beta t_1(x)\right\} \right\} \\ &= \arg\max_{\beta} \left\{-\beta t_1(x) - \log Z(\beta)\right\}\end{aligned}$$

- However, $\log Z(\beta)$ has an intractable form

$$\log Z(\beta) = \log \sum_x \exp\left\{-\beta t_1(x)\right\}$$

- Partition function can not be computed.

Coding Method/Maximum Pseudolikelihood[15, 16]



- Assume a 4 point neighborhood
- Separate points into four groups or codes.
- Group (code) contains points which are conditionally independent given the other groups (codes).

$$\hat{\beta} = \arg\max_{\beta} \prod_{s \in \text{Code}_k} p(x_s | x_{\partial s})$$

- This is tractable (but not necessarily easy) to compute

Least Squares Parameter Estimation[49]

- It can be shown that for an Ising model

$$\log \frac{P\{X_s = 1 | x_{\partial s}\}}{P\{X_s = 0 | x_{\partial s}\}} = -\beta \left(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}) \right)$$

- For each unique set of neighboring pixel values, $x_{\partial s}$, we may compute
  - The observed rate of $\log \frac{P\{X_s=1|x_{\partial s}\}}{P\{X_s=0|x_{\partial s}\}}$
  - The value of $(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}))$
  - This produces a set of over-determined linear equations which can be solved for β.
- This least squares method is easily implemented.

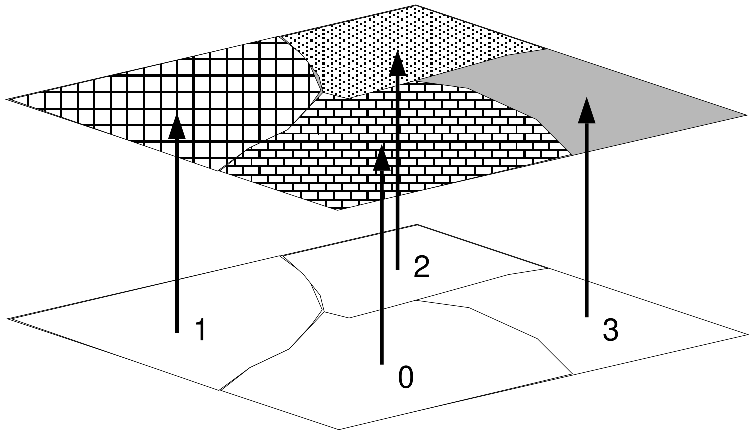
Theoretical Results in Parameter Estimation for MRF's

- Inconsistency of ML estimate for Ising model[130, 131]
  - Caused by critical temperature behavior.
  - Single sample of Ising model cannot distinguish between high β with mean 1/2, and low β with large mean.
  - Not identifiable
- Consistency of maximum pseudolikelihood estimate[69]
  - Requires an identifiable parameterization.

Application of MRF's to Segmentation

- Topics to be covered:
  - The Model
  - Bayesian Estimation
  - MAP Optimization
  - Parameter Estimation
  - Other Approaches

Bayesian Segmentation Model



- Discrete MRF is used to model the segmentation field.
- Each class is represented by a value $X_s \in \{0, \cdots, M-1\}$
- The joint probability of the data and segmentation is

$$P\{Y \in dy, X = x\} = p(y|x)p(x)$$

  where

  - $p(y|x)$ is the data model
  - $p(x)$ is the segmentation model

Bayes Estimation

- $C(x, X)$ is the cost of guessing x when X is the correct answer.
- $\hat{X}$ is the estimated value of X.
- $E[C(\hat{X}, X)]$ is the expected cost (risk).
- Objective: Choose the estimator $\hat{X}$ which minimizes $E[C(\hat{X}, X)]$.

Maximum *A Posteriori* (MAP) Estimation

- Let $C(x, X) = \delta(x \neq X)$
- Then the optimum estimator is given by

$$
\begin{aligned}
\hat{X}_{MAP} &= \arg\max_x p_{x|y}(x|Y) \\
&= \arg\max_x \log \frac{p_{y,x}(Y, x)}{p_y(Y)} \\
&= \arg\max_x \{\log p(Y|x) + \log p(x)\}
\end{aligned}
$$

- Advantage:
  - Can be computed through direct optimization
- Disadvantage:
  - Cost function is unreasonable for many applications

Maximizer of the Posterior Marginals (MPM) Estimation[116]

- Let $C(x, X) = \sum_{s \in S} \delta(x_s \neq X_s)$
- Then the optimum estimator is given by

$$\hat{X}_{MPM} = \arg\max_{x_s} p_{x_s|Y}(x_s|Y)$$

- Compute the most likely class for each pixel
- Method:
  - Use simulation method to generate samples from $p_{x|y}(x|y)$.
  - For each pixel, choose the most frequent class.
- Advantage:
  - Minimizes number of misclassified pixels
- Disadvantage:
  - Difficult to compute

MAP Optimization for Segmentation

- Assume the data model

$$p_{y|x}(y|x) = \prod_{s \in S} p(y_s|x_s)$$

- And the prior model (Ising model)

$$p_x(x) = \frac{1}{Z'} \exp\{-\beta t_1(x)\}$$

- Then the MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{ -\log p_{y|x}(y|x) + \beta t_1(x) \right\}$$

- This optimization problem is very difficult

Iterated Conditional Modes [16]

- The problem:

$$\hat{x}_{MAP} = \arg\min_x \left\{ - \sum_{s \in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x) \right\}$$

- Iteratively minimize the function with respect to each pixel, x_s.

$$\hat{x}_s = \arg\min_{x_s} \left\{ - \log p_{y_s|x_s}(y_s|x_s) + \beta v_1(x_s|x_{\partial s}) \right\}$$

- This converges to a local minimum in the cost function

Simulated Annealing [68]

- Consider the Gibbs distribution

$$\frac{1}{Z}\exp\left\{-\frac{1}{T}U(x)\right\}$$

where

$$U(x) = \sum_{s\in S} \log p_{y_s|x_s}(y_s|x_s) + \beta t_1(x)$$

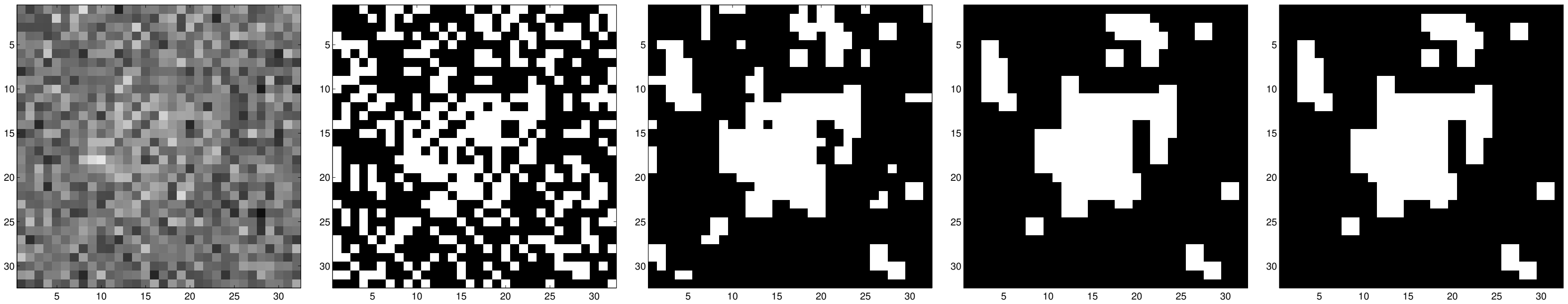
- As $T \rightarrow 0$, the distribution becomes clustered about $\hat{x}_{MAP}$.
- Use simulation method to generate samples from distribution.
- Slowly let $T \rightarrow 0$.
- If $T_k = \frac{T_1}{1+\log k}$ for iteration k, the the simulation converges to $\hat{x}_{MAP}$ almost surely.
- Problem: This is very slow!

Multiscale MAP Segmentation

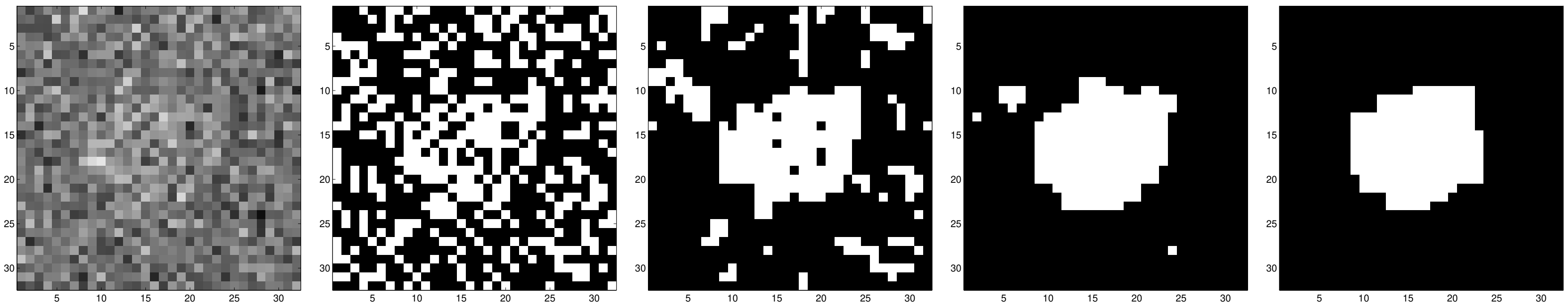
- Renormalization theory[72]
  - Theoretically results in the exact MAP segmentation
  - Requires the computation of intractable functions
  - Can be implemented with approximation
- Multiscale resolution segmentation[23]
  - Performs ICM segmentation in a coarse-to-fine sequence
  - Each MAP optimization is initialized with the solution from the previous coarser resolution
  - Used the fact that a discrete MRF constrained to be block constant is still a MRF.
- Multiscale Markov random fields[97]
  - Extended MRF to the third dimension of scale
  - Formulated a parallel computational approach

Segmentation Example

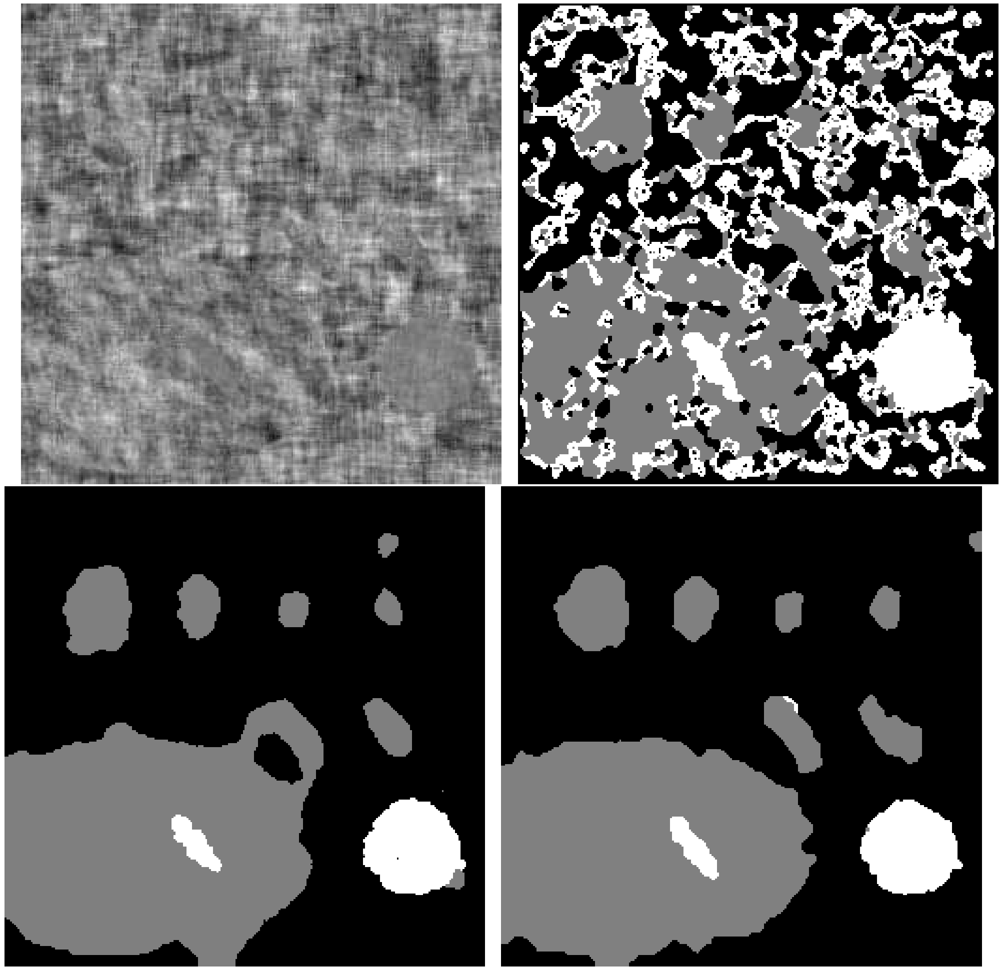
- Iterated Conditional Modes (ICM): ML ; ICM 1; ICM 5; ICM 10



- Simulated Annealing (SA): ML ; SA 1; SA 5; SA 10



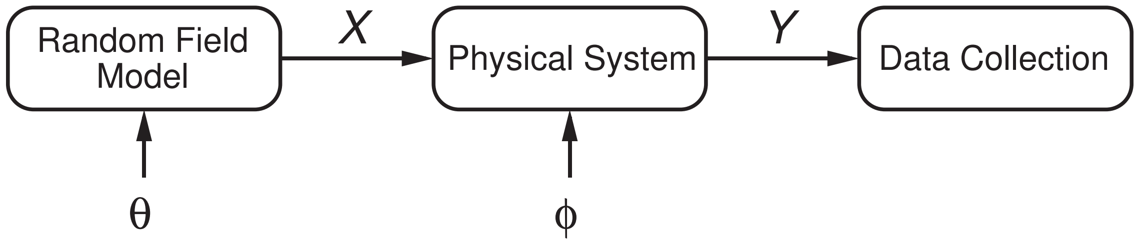
Texture Segmentation Example



a	b
c	d

a) Synthetic image with 3 textures b) ICM - 29 iterations c) Simulated Annealing - 100 iterations d) Multiresolution - 7.8 iterations

Parameter Estimation



- Question: How do we estimate θ from Y?
- Problem: We don't know X!
- Solution 1: Joint MAP estimation [104]

$$(\hat{\theta}, \hat{x}) = \arg\max_{\theta, x} p(y, x|\theta)$$

  – Problem: The solution is biased.

- Solution 2: Expectation maximization algorithm [9, 70]

$$\hat{\theta}^{k+1} = \arg\max_{\theta} E[\log p(Y, X|\theta)|Y = y, \theta^k]$$

  – Expectation may be computed using simulation techniques or mean field theory.

Other Approaches to using Discrete MRFs

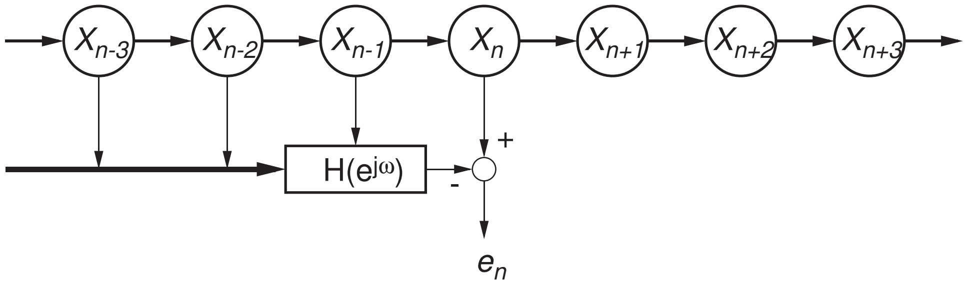
- Dynamic programming does not work in 2-D, but a number of researchers have formulated approximate recursive solutions to MAP estimation[48, 169].
- Mean field theory has also been studied as a method for computing the MPM estimate[176].

Gaussian Random Process Models

- Topics to be covered:
  - Autoregressive (AR) models
  - Simultaneous Autoregressive (SAR) models
  - Gaussian MRF's
  - Generalization to 2-D

Autoregressive (AR) Models

$$e_n = x_n - \sum_{k=1}^{\infty} x_{n-k} h_k$$



- $H(e^{j\omega})$ is an optimal predictor $\Rightarrow e(n)$ is white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{-\frac{1}{2} x^t \mathbf{A}^t \mathbf{A} x\right\}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$$

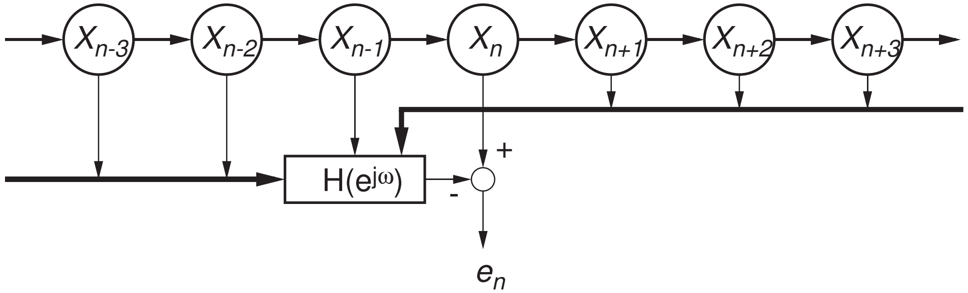
$$Z = (2\pi)^{N/2} |\mathbf{A}|^{-1} = (2\pi)^{N/2}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Simultaneous Autoregressive (SAR) Models[95, 94]

$$e_n = x_n - \sum_{k=1}^{\infty}(x_{n-k} - x_{n+k})h_k$$



- $e(n)$ is white noise $\Rightarrow H(e^{j\omega})$ is **not** an optimal non-causal predictor.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z}\exp\left\{-\frac{1}{2}x^t\mathbf{A}^t\mathbf{A}x\right\}$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & & -h_{m-n} \\ & \ddots & \\ -h_{n-m} & & 1 \end{bmatrix}$$

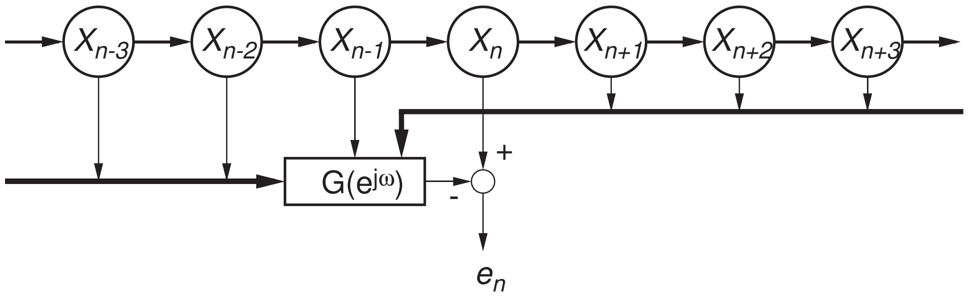
$$Z = (2\pi)^{N/2}|\mathbf{A}|^{-1} \approx (2\pi)^{N/2}\exp\left\{-\frac{N}{2\pi}\int_{-\pi}^{\pi}\log|1 - H(e^{j\omega})|d\omega\right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{|1 - H(e^{j\omega})|^2}$$

Conditional Markov (CM) Models (i.e. MRF's)[95, 94]

$$e_n = x_n - \sum_{k=1}^{\infty} (x_{n-k} - x_{n+k}) g_k$$



- $G(e^{j\omega})$ is an optimal non-causal predictor $\Rightarrow$ $e(n)$ is **not** white noise.
- The density for the N point vector X is given by

$$p_x(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{2} x^t \mathbf{B} x \right\}$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & & -g_{m-n} \\ & \ddots & \\ -g_{n-m} & & 1 \end{bmatrix}$$

$$Z = (2\pi)^{N/2} |\mathbf{B}|^{-1/2} \approx (2\pi)^{N/2} \exp\left\{ -\frac{N}{4\pi} \int_{-\pi}^{\pi} \log(1 - G(e^{j\omega})) d\omega \right\}$$

- The power spectrum of X is

$$S_x(e^{j\omega}) = \frac{\sigma_e^2}{1 - G(e^{j\omega})}$$

Generalization to 2-D

- Same basic properties hold.
- Circulant matrices become circulant block circulant.
- Toeplitz matrices become Toeplitz block Toeplitz.
- SAR and MRF models are more important in 2-D.

Non-Gaussian Continuous State MRF's

- Topics to be covered:
  - Quadratic functions
  - Non-Convex functions
  - Continuous MAP estimation
  - Convex functions

Why use Non-Gaussian MRF's?

- Gaussian MRF's do not model edges well.
- In applications such as image restoration and tomography, Gaussian MRF's either
  - Blur edges
  - Leave excessive amounts of noise

Gaussian MRF's

- Gaussian MRF's have density functions with the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\sum_{s \in S} a_s x_s^2 - \sum_{\{s,r\} \in C} b_{sr} |x_s - x_r|^2 \right\}$$

- We will assume $a_s = 0$.
- The terms $|x_s - x_r|^2$ penalize rapid changes in gray level.
- MAP estimate has the form

$$\hat{x} = \arg\min_x \left\{ -\log p(y|x) + \sum_{\{s,r\} \in C} b_{sr} |x_s - x_r|^2 \right\}$$

- **Problem:** Quadratic function, $|\cdot|^2$, excessively penalizes image edges.

Non-Gaussian MRF's Based on Pair-Wise Cliques

- We will consider MRF's with pair-wise cliques

$$p(x) = \frac{1}{Z} \exp\left\{ - \sum_{\{s,r\} \in C} b_{sr} \rho\left(\frac{x_s - x_r}{\sigma} \right) \right\}$$

$|x_s - x_r|$ - is the change in gray level.

σ - controls the gray level variation or scale.

$\rho(\Delta)$:

– Known as the potential function.

– Determines the cost of abrupt changes in gray level.

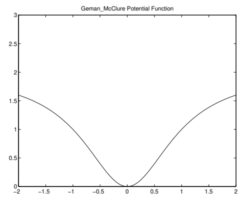
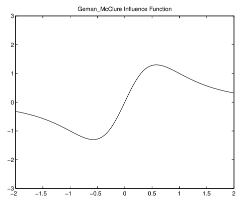
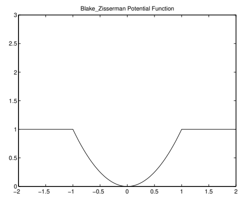
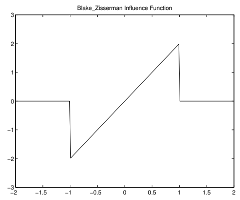
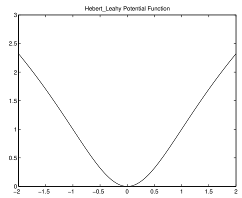
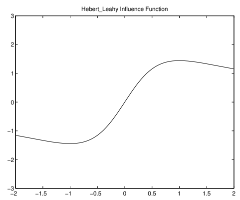
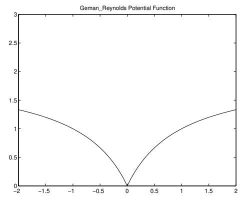
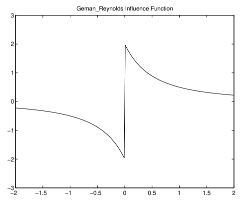
– $\rho(\Delta) = |\Delta|^2$ is the Gaussian model.

$\rho'(\Delta) = \frac{d\rho(\Delta)}{d\Delta}$:

– Known as the influence function from "M-estimation"[139, 85].

– Determines the attraction of a pixel to neighboring gray levels.

Non-Convex Potential Functions

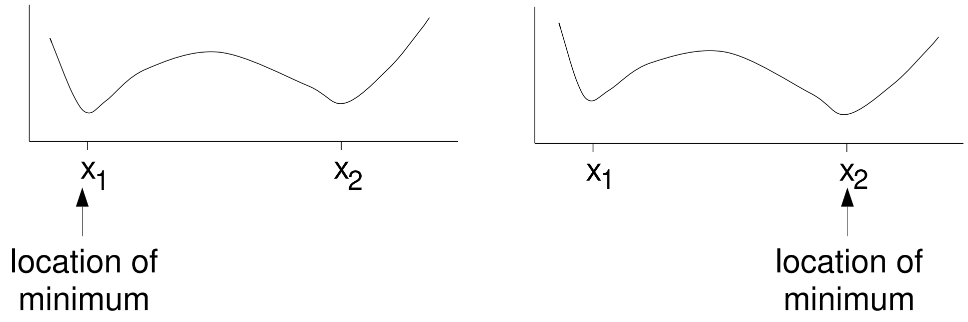
Authors	$\rho(\Delta)$	Ref.	Potential func.	Influence func.
Geman and McClure	$\frac{\Delta^2}{1+\Delta^2}$	[70, 71]		
Blake and Zisserman	$\min\left\{\Delta^2, 1\right\}$	[20, 19]		
Hebert and Leahy	$\log\left(1+\Delta^2\right)$	[81]		
Geman and Reynolds	$\frac{\|\Delta\|}{1+\|\Delta\|}$	[66]		

Properties of Non-Convex Potential Functions

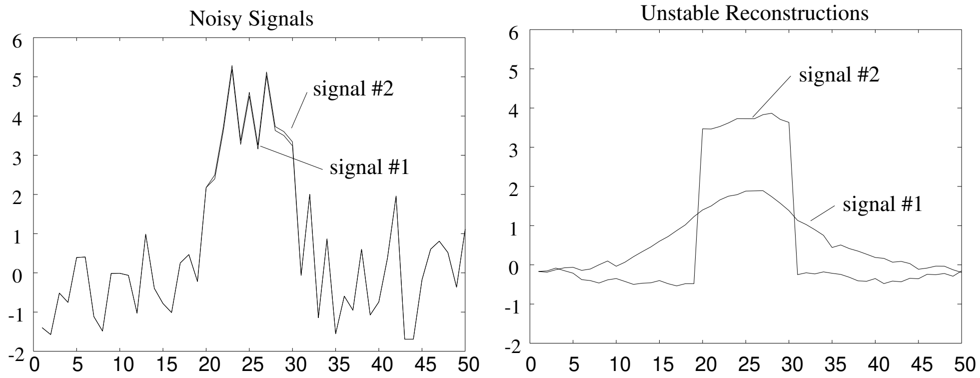
- Advantages
  - Very sharp edges
  - Very general class of potential functions
- Disadvantages
  - Difficult (impossible) to compute MAP estimate
  - Usually requires the choice of an edge threshold
  - **MAP estimate is a discontinuous function of the data**

Continuous (Stable) MAP Estimation[25]

- Minimum of non-convex function can change abruptly.

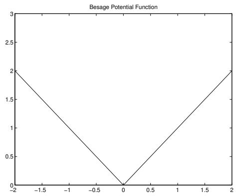
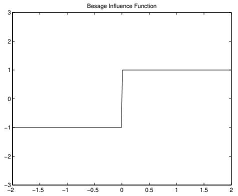
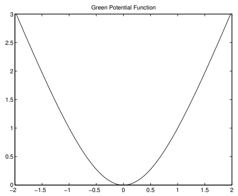
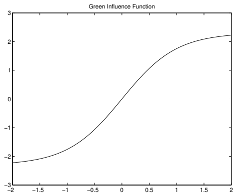


- Discontinuous MAP estimate for Blake and Zisserman potential.



- Theorem:[25] - If the log of the posterior density is **strictly convex**, then the MAP estimate is a continuous function of the data.

Convex Potential Functions

<table>
<tr><th>Authors(Name)</th><th>$\rho(\Delta)$</th><th>Ref.</th><th>Potential func.</th><th>Influence func.</th></tr>
<tr><td>Besag</td><td>$|\Delta|$</td><td>[17]</td><td>
</td><td>
</td></tr>
<tr><td>Green</td><td>$\log\cosh\Delta$</td><td>[75]</td><td>
</td><td>
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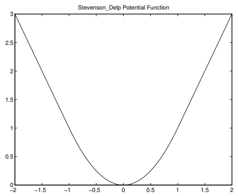
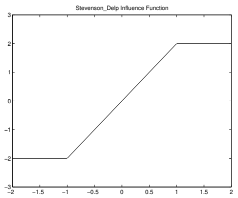
<tr><td>Stevenson and Delp (Huber function)</td><td>$\min\{|\Delta|^2, 2|\Delta|-1\}$</td><td>[155]</td><td>
</td><td>
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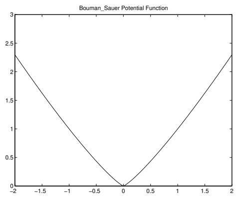
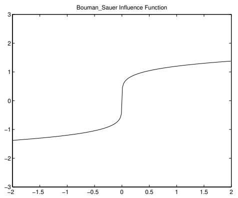
<tr><td>Bouman and Sauer (Generalized Gaussian MRF)</td><td>$|\Delta|^p$</td><td>[25]</td><td>
</td><td>
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</table>

Properties of Convex Potential Functions

- Both $\log\cosh(\Delta)$ and Huber functions
  - Quadratic for $|\Delta| << 1$
  - Linear for $|\Delta| >> 1$
  - Transition from quadratic to linear determines edge threshold.
- Generalized Gaussian MRF (GGMRF) functions
  - Include $|\Delta|$ function
  - Do not require an edge threshold parameter.
  - Convex and differentable for $p > 1$.

Parameter Estimation for Continuous MRF's

- Topics to be covered:
  - Estimation of scale parameter, σ
  - Estimation of temperature, T, and shape, p

ML Estimation of Scale Parameter, σ, for Continuous MRF's [26]

- For any continuous state Gibbs distribution

$$p(x) = \frac{1}{Z(\sigma)} \exp\left\{-U(x/\sigma)\right\}$$

  the partition function has the form

$$Z(\sigma) = \sigma^N Z(1)$$

- Using this result the ML estimate of σ is given by

$$\left.\frac{\sigma}{N}\frac{d}{d\sigma}U(x/\sigma)\right|_{\sigma=\hat{\sigma}} - 1 = 0$$

- This equation can be solved numerically using any root finding method.

ML Estimation of σ for GGMRF's [108, 26]

- For a Generalized Gaussian MRF (GGMRF)

$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ -\frac{1}{p\sigma^p} U(x) \right\}$$

where the energy function has the property that for all $\alpha > 0$

$$U(\alpha x) = \alpha^p U(x)$$

- Then the ML estimate of σ is

$$\hat{\sigma} = \left(\frac{1}{N} U(x) \right)^{(1/p)}$$

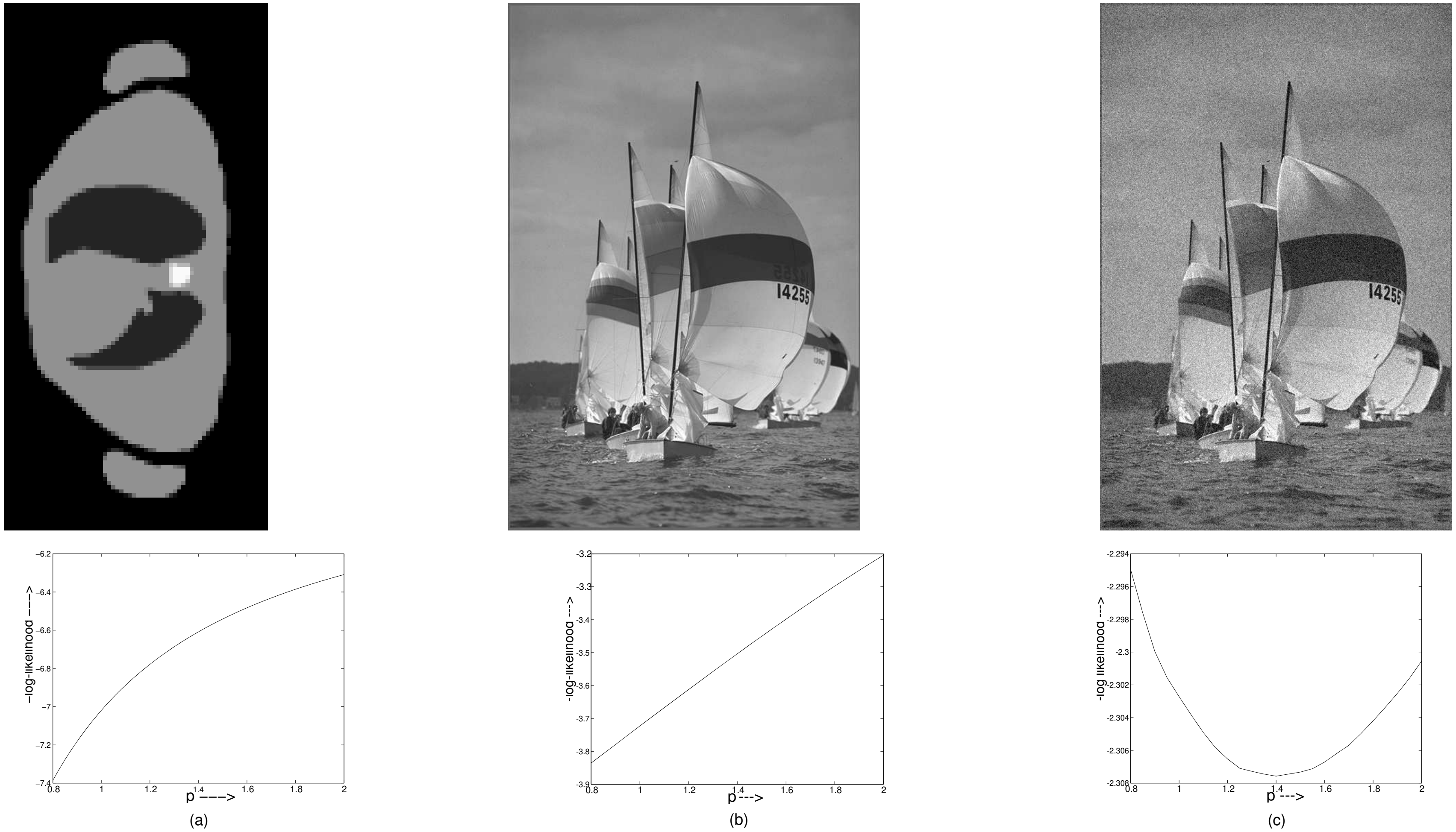
- Notice for that for the i.i.d. Gaussian case, this is

$$\hat{\sigma} = \sqrt{\frac{1}{N} \sum_s |x_s|^2}$$

Estimation of Temperature, T, and Shape, p, Parameters

- ML estimation of T[71]
  - Used to estimate T for any distribution.
  - Based on "off line" computation of log partition function.
- Adaptive method [133]
  - Used to estimate p parameter of GGMRF.
  - Based on measurement of kurtosis.
- ML estimation of p[145, 144]
  - Used to estimate p parameter of GGMRF.
  - Based on "off line" computation of log partition function.

Example Estimation of p Parameter



- ML estimation of p for (a) transmission phantom (b) natural image (c) image corrupted with Gaussian noise. The plot below each image shows the corresponding negative log-likelihood as a function of p. The ML estimate is the value of p that minimizes the plotted function.

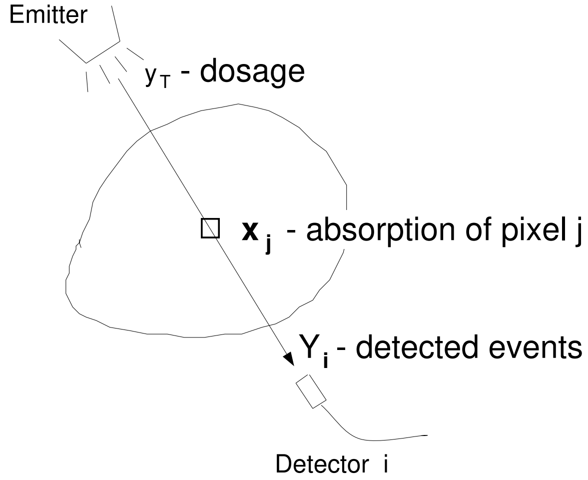
Application to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

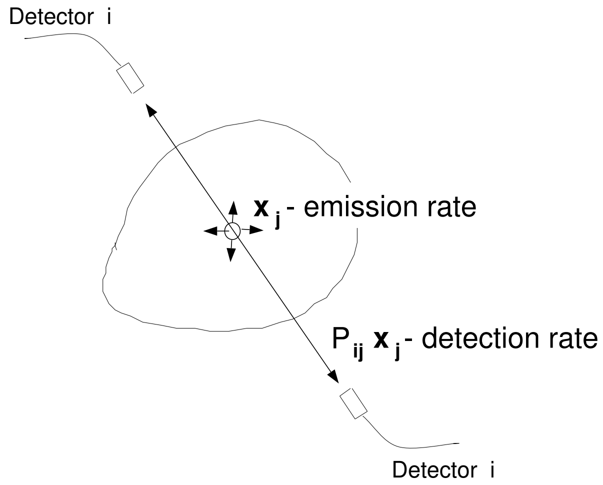
The Tomography Problem

- Recover image cross-section from integral projections

- Transmission problem



- Emission problem



Statistical Data Model[27]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x \ + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\Sigma_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$
\begin{aligned}
\hat{x} &= \arg\max_{x} p(x|y) \\
&= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\, \rho(x_k - x_j) \right\}
\end{aligned}
$$

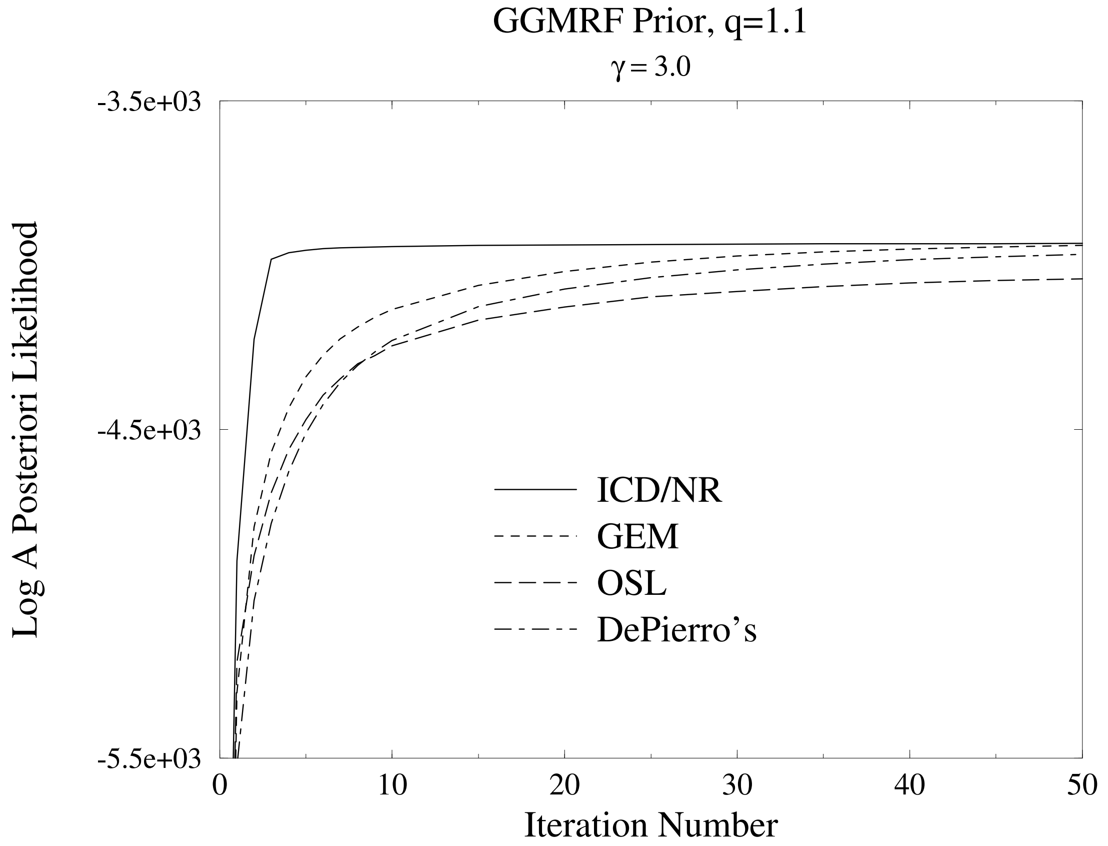
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[151, 107]
  - MAP reconstruction[81, 75, 84]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[59]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[45]
  - ICM iterations (also known as ICD and Gauss-Seidel)[27]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

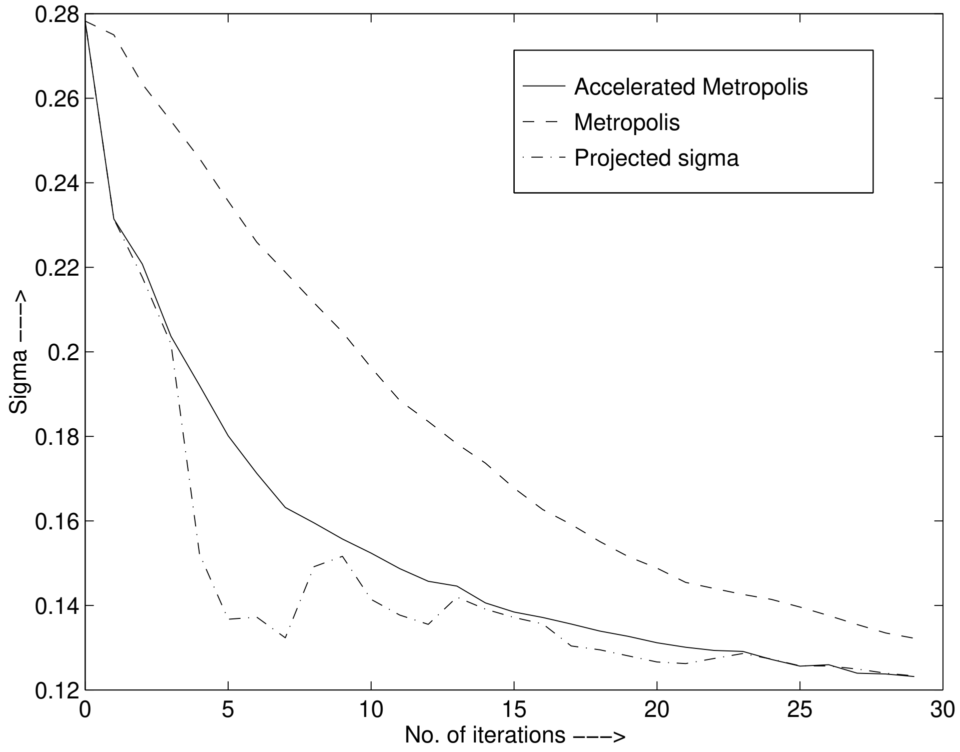
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{ \frac{1}{p\sigma^p} U(x) \right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned} \sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{ \log p(X|\sigma) | Y = y, \sigma^{(k)} \right\} \\ &= \left(E\left\{ \frac{1}{N} U(X) | Y = y, \sigma^{(k)} \right\} \right)^{1/p} \end{aligned}$$

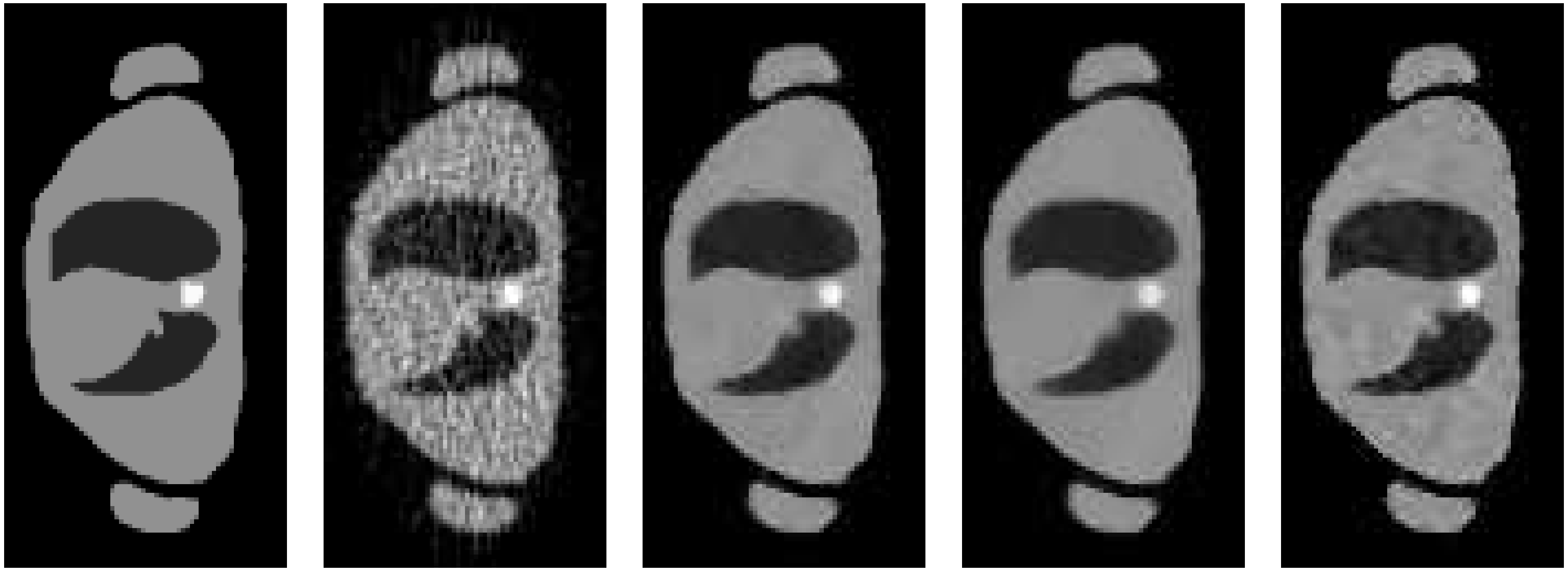
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

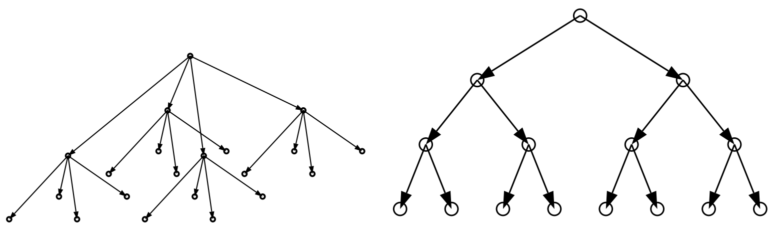


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[12, 5, 111]
  - Discrete models[29, 111]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

Multiscale Stochastic Models for Continuous State Estimation

- Theory of 1-D systems can be extended to multiscale trees[6, 7].
- Can be used to efficiently estimate optical flow[111].
- These models can approximate MRF's[112].
- The structure of the model allows exact calculation of log likelihoods for texture segmentation[113].

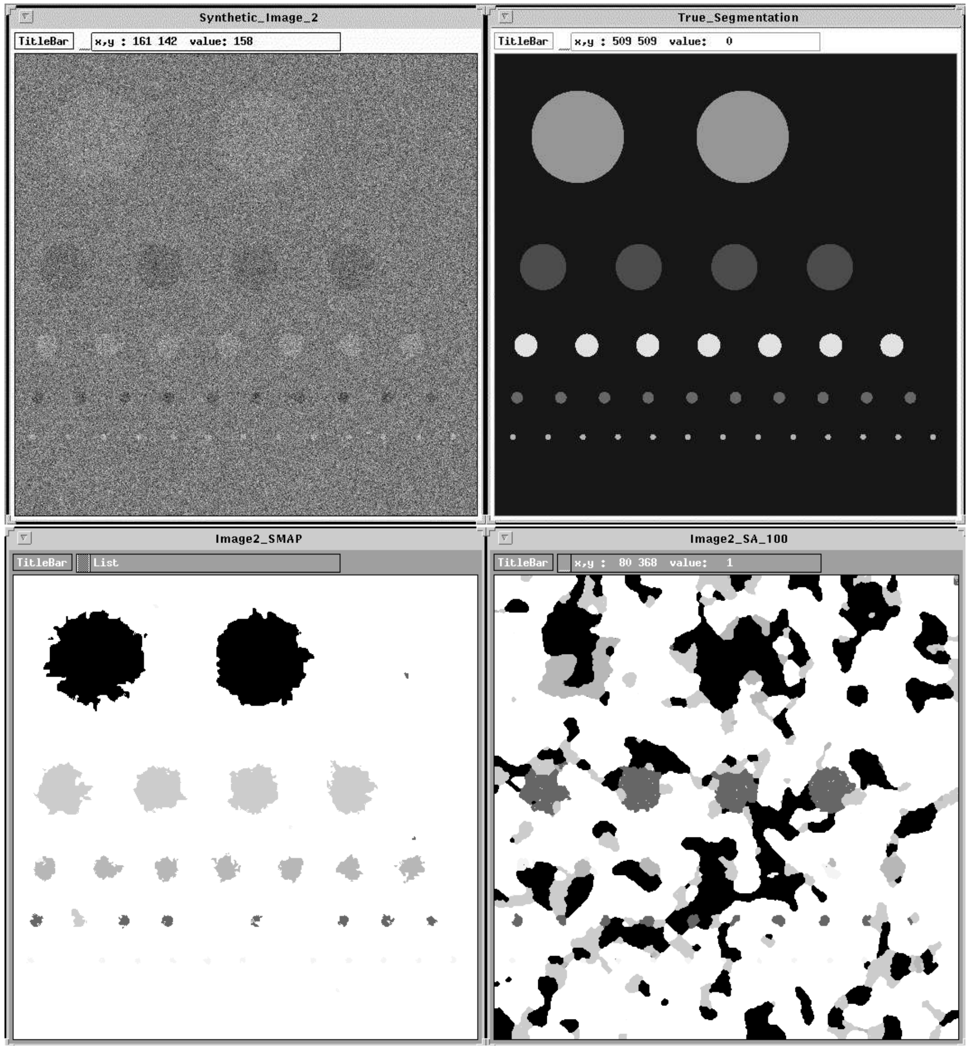
Multiscale Stochastic Models for Segmentation[29]

- Multiscale model results in non-iterative segmentation
- Sequential MAP (SMAP) criteria minimizes size of largest misclassification.
- Computational comparison

	Replacements per pixel				
	SMAP	SMAP + par. est.	SA 500	SA 100	ICM
image1	1.33	3.13	504	105	28
image2	1.33	3.55	506	108	28
image3	1.33	3.14	505	104	10

Segmentation of Synthetic Test Image

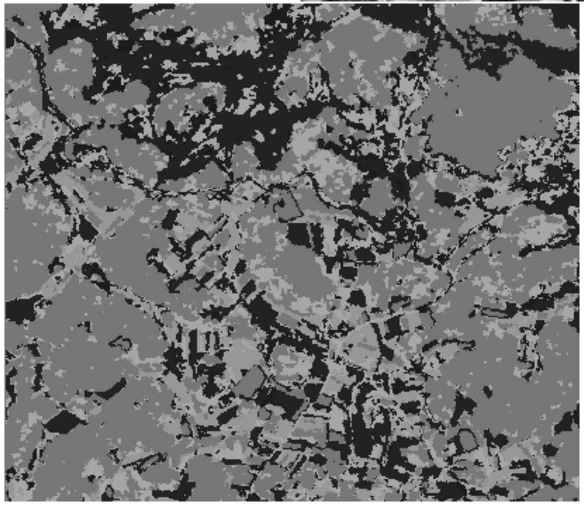
Synthetic Image | Correct Segmentation

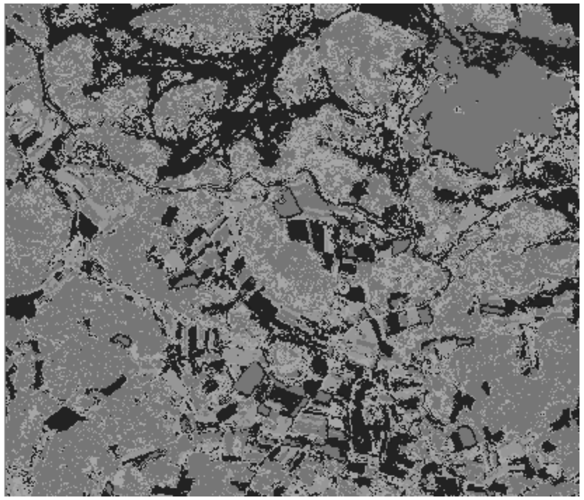


SMAP | 100 Iterations of SA

Multispectral Spot Image Segmentation

SPOT image





SMAP

Maximum Likelihood

High Level Image Models

- MRF's have been used to
  - model the relative location of objects in a scene[119].
  - model relational constraints for object matching problems[109].
- Multiscale stochastic models
  - have been used to model complex assemblies for automated inspection[166].
  - have been used to model 2-D patterns for application in image search[154].

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