A Model-Based Image Reconstruction Algorithm with Simultaneous Beam Hardening Correction for X-Ray CT

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Abstract—Beam hardening is a well-known effect in X-ray CT scanning that is caused by the interaction of a broad polychromatic source spectrum with energy-dependent material attenuation. If the scanned object only consists of a single material, the beam hardening effect can be corrected by sinogram pre-correction techniques. However, when multiple materials are present, it becomes much more difficult to fully compensate for this distortion, and in general the beam hardening can contribute to reconstruction artifacts such as cupping and streaking.

In this paper, we present a novel model-based iterative reconstruction algorithm that incorporates beam hardening correction (MBIR-BHC). Unlike most correction algorithms, which require knowledge of the X-ray spectrum or mass attenuation functions, the MBIR-BHC algorithm works by simultaneously reconstructing the image and estimating the beam-hardening function. The method is based on the assumption that the object is formed by a combination of two distinct materials that can be separated based on their densities. We formulate a poly-energetic X-ray forward model using a polynomial function of two material projections: one for the low density material and one for the high. We then develop an alternating minimization algorithm for jointly estimating the reconstructed image, the material segmentation, and the coefficients of the two component polynomial that models the beam-hardening function. With this approach, the spectrum and mass attenuation functions are not needed in advance, and the correction is adapted to the dataset being reconstructed.

We examine the performance of the proposed algorithm using both simulated and real datasets. Results indicate that the MBIR-BHC algorithm significantly reduces several reconstruction artifacts without advanced knowledge of the X-ray spectrum and material properties.

Index Terms—X-ray CT, model-based iterative reconstruction (MBIR), beam hardening correction, poly-energetic

I. INTRODUCTION

X-Ray computed tomography (CT) is a widely used imaging modality that depends on the reconstruction of material cross-sections from line integrals of X-ray density. Typically, the required line integrals are obtained by assuming that the X-rays are attenuated exponentially as predicted by Beer-Lambert’s law [1]. However, this approximation only holds when the X-ray source is monochromatic. When the X-ray source has a broad spectrum, low-energy photons are typically attenuated more rapidly than high-energy photons; so the beam shifts toward higher energies (i.e., is “hardened”) as it passes through the material. The total attenuation is formed by the superposition of weighted exponentials, resulting in the so-called beam-hardening effect [2], [3]. In practice, beam hardening can contribute to cupping and streaking artifacts in the reconstructed images [2]–[7].

Various algorithms have been proposed to address the beam hardening effect [8]–[33]. One early approach was to pre-filter the X-ray beam [19] by placing a thin metal plate between the X-ray source and the objects, so as to pre-attenuate the low-energy photons. While such a method is able to narrow the X-ray spectrum and, therefore, reduce the beam hardening effect, it also lowers the detected signal to noise ratio (SNR). Another approach is dual energy scanning [11], [12]. These methods work by reconstructing two material-independent density maps from low and high-energy X-ray measurements. While this technique can fully account for beam-hardening, it requires two spectrally distinct projection measurements, which is generally much more complex and expensive.

Algorithmic correction is perhaps the most common approach to beam-hardening correction. One widely-used algorithmic correction is linearization, or polynomial pre-correction [2], [8]–[10]. Pre-correction techniques are based on the assumption that the object is made from a single material, such as water. However, when multiple materials are present, it is not possible to fully compensate for the beam hardening distortion using a single pre-correction function. Alternatively, iterative post-processing techniques are also widely used when the object being scanned is composed of two known materials that can be easily segmented [13]–[17]. These methods correct the sinogram using knowledge of the two materials’ mass attenuation functions, the X-ray source spectrum, and an approximate spatial segmentation into the two materials. In medical imaging applications, it may be reasonable to assume that the object being scanned can be approximately segmented into two known materials, (e.g., soft issue and bone) and the X-ray spectrum is known; so these iterative post-processing techniques can be effective [13], [15], [16]. However, in more general CT applications, such as non-destructive evaluation or security scanning, the objects being scanned may be composed of a number of unknown materials. So the assumptions of iterative post-processing algorithms are violated, and the methods are not straightforward to apply.

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Recent results in model-based iterative reconstruction (MBIR) have demonstrated its ability to improve the reconstructed image quality [34]–[38]. MBIR algorithms typically work by first formulating an overall objective function which incorporates statistical models of both the forward acquisition processes and the objects being reconstructed. The resulting objective functions are then minimized using iterative optimization methods. Several methods have also been developed in order to address the problem of beam hardening in the context of iterative reconstruction [22]–[25]. De Man et al. [22] proposed an iterative method which incorporated the knowledge of the known X-ray spectrum into the reconstruction process in order to account for the beam hardening effect. Elbakri et al. [23], [24] developed an iterative reconstruction method based on the idea of material decomposition. The joint beam hardening correction polynomial for a set of predetermined basis materials is pre-calculated and tabulated, and is utilized during the iterative reconstruction. Srivastava et al. [26] and Abella et al. [30] extended Elbakri’s approach. Their strategy is to design functionals with tuning parameters so that it can map the non-linear effect of two materials into one equivalent material. Therefore, only a single material beam hardening correction polynomial is required. However, all these methods require some additional knowledge of the system forward projection matrix whose entries represent the contribution of the i-th projection to the j-th pixel. For each projection, the standard CT projection measurement is generated by

\[ Y_i = -\log \frac{\lambda_i}{\lambda_{T,i}} \]

where \( \lambda_{T,i} \) is the expected photon intensity in an air-calibration scan for the i-th projection, given by

\[ \lambda_{T,i} = \int_{\mathbb{R}} S_i(\xi) d\xi. \]

Denote the normalized energy spectrum as \( \tilde{S}_i \) given by

\[ \tilde{S}_i(\xi) = \frac{S_i(\xi)}{\lambda_{T,i}}. \]

We will assume that \( \tilde{S}_i(\xi) \) is the same for all the projections; so the projection index i can be dropped in \( \tilde{S}_i \). Putting (1) through (4), the expected projection measurement can be approximated as

\[ \mathbb{E}[Y_i|\mu(\xi)] \approx \log \left( \frac{\lambda_i}{\lambda_{T,i}} \right) \]

\[ = \log \left( \int_{\mathbb{R}} \tilde{S}(\xi) e^{-\sum_j A_{i,j} \mu_j(\xi)} d\xi \right). \]

So (5) is the conventional model for the non-linear beam hardening that results from a poly-energetic X-ray beam.

Our objective will be to formulate a simple parametric model of the beam-hardening that occurs with a single, polychromatic scan of an object composed of two distinct materials, one with high density and the other with low. To do this, we first define \( x_j \) to be the weighted average of the linear attenuation coefficient of the j-th pixel with respect to the energy spectrum

\[ x_j \triangleq \int_{\mathbb{R}} \tilde{S}(\xi) \mu_j(\xi) d\xi \]

The paper is organized as follows. Section II presents the poly-energetic X-ray model and formulates the problem of joint reconstruction and correction. Section III describes the alternating optimization. Section IV shows the experimental results on the simulated and real data to demonstrate the improvement achieved by MBIR-BHC as compared to traditional methods. Finally, Section V concludes the discussion.

II. POLY-ENERGETIC X-RAY MODEL AND STATISTICAL APPROACH FOR RECONSTRUCTION

A. Poly-energetic Model for X-ray CT

Let \( \mu(\xi) \in \mathbb{R}^N \) be the vector whose the j-th entry \( \mu_j(\xi) \) is the energy-dependent linear attenuation coefficient of the j-th pixel. The received photon intensity of the i-th projection, denoted by \( \lambda_i \), can be modeled as a Poisson random variable with the mean given by

\[ \mathbb{E}[\lambda_i|\mu(\xi)] = \int_{\mathbb{R}} S_i(\xi) e^{-\sum_j A_{i,j} \mu_j(\xi)} d\xi \]

where \( S_i(\xi) \) is the source-detector energy spectrum, and A is the system forward projection matrix whose entries represent the contribution of the i-th projection to the j-th pixel. For each projection, the standard CT projection measurement \( Y_i \) is generated by

\[ Y_i = -\log \frac{\lambda_i}{\lambda_{T,i}} \]

\[ \lambda_{T,i} = \int_{\mathbb{R}} S_i(\xi) d\xi. \]
Using this definition, we can rewrite the energy-dependent linear attenuation coefficient of the $j$-th pixel as

$$
\mu_j(\mathcal{E}) = x_j r_j(\mathcal{E})
$$

(7)

where $r_j(\mathcal{E})$ is the absorption spectrum of the $j$-th pixel, given by

$$
r_j(\mathcal{E}) = \frac{\mu_j(\mathcal{E})}{x_j} = \frac{\mu_j(\mathcal{E})}{\int_{\mathcal{E}} S(\mathcal{E}) \mu_j(\mathcal{E}) d\mathcal{E}}.
$$

(8)

Notice that in this formulation, $r_j(\mathcal{E})$ carries the energy dependency and $x_j$ only depends on the pixel location $j$. Moreover, from (8) we see that the weighted energy spectrum of $r_j(\mathcal{E})$ is normalized to 1; so for all $j$,

$$
\int_{\mathcal{R}} \tilde{S}(\mathcal{E}) r_j(\mathcal{E}) d\mathcal{E} = 1.
$$

(9)

We first consider the simple case when the scanned object only contains one absorbive material denoted by $\mathcal{M}$. The functions $r_j(\mathcal{E})$ are identical for all pixels $j$ and we have

$$
\mu_j(\mathcal{E}) = x_j r_{\mathcal{M}}(\mathcal{E})
$$

(10)

where $r_{\mathcal{M}}(\mathcal{E})$ is the absorption spectra for the material $\mathcal{M}$. Substituting (10) into (5), we may define the beam hardening function $f_{\mathcal{M}}(p_i)$ given by

$$
\begin{aligned}
\mu_j(\mathcal{E}) & = x_j r_{\mathcal{M}}(\mathcal{E}) \\
& \overset{\text{def}}{=} - \log \left( \int_{\mathcal{R}} \tilde{S}(\mathcal{E}) e^{-r_{\mathcal{M}}(\mathcal{E}) p_i} d\mathcal{E} \right)
\end{aligned}
$$

(11)

where $p_i$ is the $i$-th projection given by

$$
p_i = \sum_j A_{i,j} x_j.
$$

(12)

Let $x \in \mathbb{R}^N$ be the vector with entries $x_j$ defined in (6). Using this notation, the expected projection measurement is then given by

$$
\mathbb{E}[Y_i | x] = f_{\mathcal{M}}(p_i).
$$

(13)

So from (13), we see that the expected projection measurement $\mathbb{E}[Y_i | x]$ will be a non-linear function $f_{\mathcal{M}}$ of the projection $p_i$. Differentiating (11), we show in Appendix A that

$$
\frac{d}{dp_i} f_{\mathcal{M}}(p_i) \bigg|_{p_i=0} = 1,
$$

(14)

and therefore, we also know that

$$
\frac{d}{dy_i} f_{\mathcal{M}}^{-1}(y_i) \bigg|_{y_i=0} = 1
$$

(15)

where $y_i$ is the dummy variable for the inverse function $f_{\mathcal{M}}^{-1}$. This implies that when the value of the projection is very small (i.e., when the projection passes through a thin soft material), the beam hardening effect is negligible. In practice, since human soft tissue has energy-dependent attenuation similar to water, most current medical systems perform a beam hardening pre-correction with respect to water. More specifically, typical medical imaging systems apply a beam-hardening correction with the form $f_{\mathcal{M}}^{-1} = f_{W}^{-1}$, where the subscript W explicitly indicates water as the reference material.

Next consider the case of two materials. In this case, a single correction function can not fully compensate for the effects of beam hardening. In order to better model this case, we will assume the object is made of two distinct materials, one of low density and a second of high density. More formally, we model the absorption spectrum $r_j(\mathcal{E})$ as a convex combination of two distinct absorption spectrum given by

$$
r_j(\mathcal{E}) \overset{\text{def}}{=} (1-b_j)r_L(\mathcal{E}) + b_j r_H(\mathcal{E}),
$$

(16)

where $r_L(\mathcal{E})$ and $r_H(\mathcal{E})$ represent the absorption spectrum of the “low” and “high” density materials respectively, and $b_j$ represents the fraction of material that is of high density for the $j$-th pixel. Using this model, the linear attenuation coefficient of the $j$-th pixel can be written as

$$
\mu_j(\mathcal{E}) = x_j ((1-b_j)r_L(\mathcal{E}) + b_j r_H(\mathcal{E})).
$$

(17)

In this work, we consider only the case in which $b_j$ is binary, i.e., $b_j \in \{0,1\}$; so each pixel will be composed entirely of either low or high density materials. In order to determine the values of $b_j$, we will estimate them directly from the CT data as part of the reconstruction process. While more discrete classes could be used, this simple two-material decomposition model strikes a balance between accuracy and model simplicity.

Substituting (17) into (5), we obtain

$$
\mathbb{E}[Y_i | x] = h(p_{L,i}, p_{H,i})
$$

(18)

where $h(p_{L,i}, p_{H,i})$ is now a two dimensional beam hardening function given by

$$
h(p_{L,i}, p_{H,i}) \overset{\text{def}}{=} - \log \left( \int_{\mathcal{R}} \tilde{S}(\mathcal{E}) e^{-r_L(\mathcal{E}) p_{L,i}-r_H(\mathcal{E}) p_{H,i}} d\mathcal{E} \right)
$$

(19)

and $p_{L,i}$ and $p_{H,i}$ are now the projections of the low and high density materials given by

$$
p_{L,i} = \sum_j A_{i,j} x_j (1-b_j),
$$

(20)

and

$$
p_{H,i} = \sum_j A_{i,j} x_j b_j.
$$

(21)

So from (18), we see that with two materials, the expected projection measurement is now a non-linear function of the two dimensional projections of the two materials. Our approach will be to adaptively estimate this 2D beam hardening function during the reconstruction process. To do this, we adopt a simple polynomial parametrization of the function given by

$$
h(p_{L,i}, p_{H,i}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \gamma_{k,l} (p_{L,i})^k (p_{H,i})^l,
$$

(22)

where $\gamma_{k,l}$ are coefficients to be jointly estimated during the reconstruction.

In fact, some of the coefficients in (22) are determined by the physics; so this will simplify our problem. More specifically, if both the projections $p_{L,i}$ and $p_{H,i}$ are 0, plugging them into (19), we see that

$$
\gamma_{0,0} = h(0,0) = - \log \left( \int_{\mathcal{R}} \tilde{S}(\mathcal{E}) e^0 d\mathcal{E} \right) = 0.
$$

(23)
Also differentiating (19) with respect to \( p_{L,i} \) and \( p_{H,i} \), we obtain in Appendix A the following two relationships

\[
\begin{align*}
\gamma_{0,0} &= \frac{\partial}{\partial p_{L,i}} h(p_{L,i}, 0) \bigg|_{p_{L,i}=0} = 1 , \\
\gamma_{0,1} &= \frac{\partial}{\partial p_{H,i}} h(0, p_{H,i}) \bigg|_{p_{H,i}=0} = 1 .
\end{align*}
\]  

Table I lists the coefficients of the function \( \hat{h} \). We will refer to an \( p \)-th order model as one that includes all the unknown coefficients for \( 0 \leq k + l \leq p \).

This two-material beam-hardening model can also be used in the case when the projection measurement is pre-corrected for beam hardening of a single material. To see this, suppose the projection measurements have been pre-corrected with respect to the material \( M \) using the function \( f_{M}^{-1} \). As a result, the expected projection measurement, after pre-correction, is approximately given by

\[
E[Y_i|x] = \tilde{h}(p_{L,i}, p_{H,i})
\]  

where the 2D beam hardening function \( \tilde{h} \) is now given by

\[
\tilde{h}(p_{L,i}, p_{H,i}) = f_{M}^{-1} \left( -\log \left( \int \tilde{S}(E)e^{-r_L(E)p_{L,i} - r_H(E)p_{H,i}}dE \right) \right).
\]

Using the similar approach as in (22), we can parametrize \( \tilde{h} \) using a high-order polynomial as

\[
\tilde{h}(p_{L,i}, p_{H,i}) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \tilde{\gamma}_{k,l} (p_{L,i})^k (p_{H,i})^l
\]

where \( \tilde{\gamma}_{k,l} \) are the polynomial coefficients. Moreover, we show in Appendix A that in this case similar constraints hold with

\[
\begin{align*}
\tilde{\gamma}_{0,0} &= \tilde{h}(0, 0) = 0 , \\
\tilde{\gamma}_{1,0} &= \frac{\partial}{\partial p_{L,i}} \tilde{h}(p_{L,i}, 0) \bigg|_{p_{L,i}=0} = 1 , \\
\tilde{\gamma}_{0,1} &= \frac{\partial}{\partial p_{H,i}} \tilde{h}(0, p_{H,i}) \bigg|_{p_{H,i}=0} = 1 .
\end{align*}
\]

\[\hat{h}(p_{L,i}, 0) = f_{L}^{-1}(f_{L}(p_{L,i})) = p_{L,i} . \]

This implies that \( \tilde{\gamma}_{k,0} = 0 \) for \( k \neq 1 \). Table II lists the coefficients of the function \( \tilde{h} \).

In summary, we have shown that the two-material beam-hardening model can be used for both pre-corrected and un-corrected projection data. In both cases, the three coefficients \( \gamma_{0,0} = 0 \) and \( \gamma_{0,1} = \gamma_{1,0} = 1 \) are pre-determined. Depending on the selected model order, the set of remaining coefficients are then estimated as part of the reconstruction algorithm. In particular, for a 2-nd order model (\( p = 2 \)), the unknown coefficients to be estimated are \( \gamma_{k,l} \) for \((k,l) \in \{(1, 1), (0, 2)\} ; \) and for a 3-rd order model (\( p = 3 \)), then the unknown coefficients to be estimated are \( \gamma_{k,l} \) for \((k,l) \in \{(1, 1), (2, 1), (0, 2), (1, 2), (0, 3)\} \).

As a final remark, we will use the unified notation \( h(\cdot, \cdot) \) and \( \gamma_{k,l} \) to denote the correction polynomial and its coefficients throughout the following discussion. This is only to keep our notation simple. We will explicitly state the pre-correction information and the polynomial order we use when we present the experiment results.

\section*{B. Statistical Model and Objective Function}

Let \( x \in \mathbb{R}^N \) be the image vector, \( y \in \mathbb{R}^M \) be the vector of the projection measurements, \( b \in \{0, 1\}^N \) be the vector of the material segmentation label mask, and \( \gamma \in \mathbb{R}^K \) be the vector of the fitting coefficients \( \gamma_{k,l} \). So \( K = 2 \) if the 2-nd order model in Table II is used and \( K = 5 \) if the 3-rd order model is used, etc. We treat \( \gamma \) as the nuisance parameter and formulate the problem of simultaneous image reconstruction and beam hardening correction as the computation of the maximum a posterior (MAP) estimate given by

\[
\hat{x}, \hat{b}, \hat{\gamma} = \arg \min_{x \geq 0, b, \gamma} \{ -\log P(y|x, b, \gamma) - \log P(x, b) \}
\]

where \( P(y|x, b, \gamma) \) is the likelihood function corresponding to the X-ray forward model, and \( P(x, b) \) is the joint prior distribution over the image \( x \) and the material segmentation mask \( b \). Note that we require the image to be non-negative. The parameter vector \( \gamma \) is adaptively estimated in this framework. Such an approach has been studied previously in [40] and one may interpret it as computation of the joint maximum a posteriori (MAP) and maximum likelihood (ML) estimates of the unknown variables and the nuisance parameters, respectively.
Assuming the projection measurements $Y_i$ are conditionally independent with mean given by (19), the negative log likelihood function can be written, within a constant, as

$$- \log \mathbb{P}(y|x, b, \gamma) \approx \frac{1}{2} \sum_{i=1}^{M} w_i (y_i - h(p_i))^2$$

(34)

where $p_i = [p_{Li}, p_{Hi}]^T$, and $w_i$ is the statistical weight for the $i$-th projection, which is approximately proportional to the inverse of variance of the measurement $Y_i$. Note that taking the log of the signal in equation (19) can cause a slight mean shift in the signal since the log is concave. However, for the purposes of this paper, we will assume that means shift is negligible. Using this assumption, $w_i$ can also be computed approximately as

$$w_i = \frac{\lambda_i^2}{\lambda_i + \sigma^2_e}$$

(35)

where $\sigma^2_e$ is the variance of the additive electronic noise.

To model the joint prior over $x$ and $b$, we adopt the Markov random field (MRF) model. We want to model not only the interactions among pixels or segmentation labels themselves, but also the interactions between pixels and segmentation labels. Our model is a two-layer pairwise MRF, one for the image $x$ and the other for the material segmentation mask $b$. Nodes on one layer connect to their corresponding counterparts on the other. Figure 1 illustrates this graphical model. From this structure, we may write the joint prior distribution over $x$ and $b$ as

$$\mathbb{P}(x, b) = \frac{1}{Z} \exp \left\{ - \sum_{\{j,k\} \in C} \alpha_{j,k} \rho(x_j, x_k) - \beta \sum_{j=1}^{N} \psi(x_j, b_j) - \eta_{j,k} \phi(b_j, b_k) \right\}$$

(36)

where $C$ denotes the set of all pairwise cliques, $\rho$, $\psi$ and $\phi$ are the positive potential functions on $x$, $b$, and the interactions between them, respectively, and $Z$ denotes the partition function making the whole function a valid probability distribution [41]. The parameters $\alpha_{j,k}$, $\beta$ and $\eta_{j,k}$ are the corresponding weights for edges in the graph. We choose $\alpha_{j,k}$ and $\eta_{j,k}$ to be inversely proportional to the distance between pixel $j$ and $k$. Moreover, the scales of these parameters are empirically adjusted to balance among noise, resolution and segmentation error in the final reconstruction. We have listed the selected parameters for each dataset we test in the Result section.

For the pixel pairwise potential $\rho$, we use the $q$-generalized Gaussian MRF (q-GGMRF) potential function [34], given by

$$\rho(x_j, x_k) \approx \frac{|x_j - x_k|^p}{1 + |x_j - x_k|/c|^{p-q}}$$

(37)

where $1 \leq q \leq p = 2$. Notice that the function only depends on the pixel difference $\Delta = x_j - x_k$ and a pair of pixels with small difference results high probability. Here $c$ is a tuning parameter to balance the performance between noise reduction and edge preservation. If $|\Delta| \ll c$, $\rho(\Delta) \approx |\Delta|^p$ and if $|\Delta| \gg c$, $\rho(\Delta) \approx |\Delta|/c^q$.

The potential $\psi$, which captures the inter-layer interactions, should be chosen such that it gives high probabilities if the segmentation label correctly reflects the pixel value and gives low probabilities otherwise. We design the potential $\psi$ to be

$$\psi(x_j, b_j) \approx (x_j - T)_+ (1 - b_j) + (T - x_j)_+ b_j$$

(38)

where $\psi(x_j, b_j) = \max(x_j, 0)$ and $T$ is the user-defined threshold. This is a linear loss function which penalizes the mismatch of the pixel and its corresponding segmentation label. Specifically, when $b_j = 0$, indicating that the pixel belongs to the low density material, $\psi$ will impose a linear penalty if $x_j$ exceeds the attenuation threshold $T$. Symmetrically, a linear penalty will be imposed in the case when $x_j < T$ and $b_j = 1$.

The potential $\phi$ over the binary segmentation label should encourage the similarity of the neighboring labels. We design it to be

$$\phi(b_j, b_k) \approx 1 - \delta(b_j - b_k)$$

(39)

where $\delta(\cdot)$ is the discrete delta function taking the value 1 at 0 and 0 elsewhere.

Combining the log likelihood term of (34) and the two-layer MRF joint prior over $x$ and $b$ in (36), we obtain the overall MAP estimation problem as

$$\{\hat{x}, \hat{b}, \hat{\gamma}\} = \arg \min_{x \geq 0, b, \gamma} \left\{ \frac{1}{2} \sum_{i=1}^{M} w_i (y_i - h(p_i))^2 + \sum_{\{j,k\} \in C} \alpha_{j,k} \rho(x_j, x_k) + \sum_{\{j,k\} \in C} \eta_{j,k} \phi(b_j, b_k) + \beta \sum_{j=1}^{N} (x_j - T)_+ (1 - b_j) + (T - x_j)_+ b_j \right\}$$

(40)

We refer to (40) as the objective function in our reconstruction framework.

III. ITERATIVE BEAM HARDENING CORRECTION AND IMAGE RECONSTRUCTION

A. Estimation of the Beam Hardening Correction Polynomial

Fixing $x$ and $b$, we first attack the problem of minimizing the objective function with respect to $\gamma$. This becomes a standard
weighted least squares problem given by
\[
\hat{y} = \arg\min_{y} \frac{1}{2} \sum_{i=1}^{M} w_i (y_i - h(p_i))^2 = \arg\min_{y} \frac{1}{2} \| y - H\gamma \|_W^2 \tag{41}
\]
where \( H \in \mathbb{R}^{M \times K} \) is a matrix whose columns correspond to specific terms in the correction polynomials of all the projections in (28) and \( W = \text{diag}\{w_1, \ldots, w_M\} \). The solution can then be computed in the closed form as
\[
\hat{y} = (H^T W H)^{-1} H^T W y . \tag{42}
\]

B. Image Reconstruction as Optimization

Next, we fix \( \gamma \) and \( b \) and minimize the objective function (40) with respect to \( x \). Since \( h(p_i) \) is a polynomial function of \( p_i \), which is in turn a linear function of \( x \), (40) will be a higher order function of \( x \), rather than a simple quadratic. Thus, the function (40) can be in general difficult to solve.

We approach the optimization by applying the Newton-Raphson technique. More specifically, we replace the original optimization (40) over \( x \) with the following modified optimization
\[
\hat{x} = \arg\min_{x \geq 0} \left\{ \sum_{i=1}^{M} (d^{i)(T}(p_i - \hat{p}_i))^2 + 2(p_i - \hat{p}_i)^T Q^{i)(p_i - \hat{p}_i)) + \beta \sum_{j} ((x_j - T_j) + (1 - b_j) + (T - x_j) + b_j) + \sum_{(j,k) \in C} \alpha_{j,k} \rho(x_j - x_k) \right\} \tag{43}
\]
where \( d^{(i)} \in \mathbb{R}^2 \) and \( Q^{(i)} \in \mathbb{R}^{2 \times 2} \) are the gradient and Hessian of the function \( h^{(i)}(y_i - h(p_i))^2 \) at the point \( p_i \), given by
\[
d^{(i)} = -w_i(y_i - h(\hat{p}_i)) \nabla h(\hat{p}_i) , \tag{44}
\]
\[
Q^{(i)} = w_i \nabla h(\hat{p}_i) \nabla h(\hat{p}_i)^T - w_i(y_i - h(\hat{p}_i)) \text{Hess}(h(\hat{p}_i)) \tag{45}
\]
and \( \hat{p}_i \) is the projection vector of the previous estimate of \( \hat{x} \). The term \( \text{Hess}(h(\hat{p}_i)) \in \mathbb{R}^{2 \times 2} \) is given by
\[
\text{Hess}(h(\hat{p}_i)) = \begin{bmatrix} \frac{\partial^2}{\partial p_i^2} h(\hat{p}_i) & \frac{\partial^2}{\partial p_i \partial p_j} h(\hat{p}_i) \\ \frac{\partial^2}{\partial p_j \partial p_i} h(\hat{p}_i) & \frac{\partial^2}{\partial p_j^2} h(\hat{p}_i) \end{bmatrix} \bigg|_{p_i = \hat{p}_i} . \tag{46}
\]

By doing this, we essentially reduce (40) to (43), which is a quadratic function plus the remaining prior terms. The same approach has been used in other non-quadratic optimization problems such as [42], [43]. Such a quadratic approximation is generally not an upper bound, which means convergence of the algorithm is not guaranteed without further innovations. However, this approximation is effective and in practice, as we will show in the Results section, we have empirically observed that the resulting updates consistently reduce the objective function of (40). The overall algorithm works by first constructing the quadratic approximation (43) using the previous projections \( \hat{p}_i \) and optimizing (43) to obtain the next estimate of \( \hat{x} \). The projections \( \hat{p}_i \) are then updated and are used in the next iteration.

There are a number of techniques that can be applied to optimize the quadratic approximation (43). We choose the iterative coordinate descent (ICD) algorithm. The ICD algorithm updates each pixel in sequence with the other pixels fixed until convergence. Here we present a sketch of derivations of the pixel update and more detailed derivations can be found in Appendix B. Following the similar strategy in [35], the \( j \)-th pixel update can be computed by solving the 1D optimization given by
\[
x_j = \arg\min_{u \geq 0} \left\{ \theta_1 (u - \bar{x}_j) + \frac{1}{2} \theta_2 (u - \bar{x}_j)^2 + \beta ((u - T_j)(1 - b_j) + (T - u) + b_j) + \sum_{k \in \partial j} \alpha_{j,k} (u - x_k)^2 \right\} . \tag{47}
\]
where \( \bar{x}_j \) denotes the previous value of the pixel, \( \partial j \) represents the set of neighbors of the \( j \)-th pixel, and the coefficient \( \alpha_{j,k} \) is given by
\[
\alpha_{j,k} = \rho(\bar{x}_j - x_k) / 2(\bar{x}_j - x_k) . \tag{48}
\]
Here \( \rho'(\cdot) \) is the first derivative of \( \rho(\cdot) \) and \( \theta_1 \) and \( \theta_2 \) are the first and second derivatives of the first term in (43) with respect to \( x_j \), given by
\[
\theta_1 = \sum_{i=1}^{M} \left( (d^{(i)} + (p_i - \hat{p}_i)) Q^{(i)} [A_{i,j}(1 - b_j)]_1 \right) \tag{49}
\]
\[
\theta_2 = \sum_{i=1}^{M} [A_{i,j}(1 - b_j)]_1 Q^{(i)} [A_{i,j}(1 - b_j)]_1 . \tag{50}
\]
To obtain the optimal solution to (47), we define two quantities \( D_1 \) and \( D_2 \) as
\[
D_1 = \theta_1 - \theta_2 \bar{x}_j - 2 \sum_{k \in \partial j} \alpha_{j,k} x_k , \tag{51}
\]
\[
D_2 = \theta_2 + 2 \sum_{k \in \partial j} \alpha_{j,k} . \tag{52}
\]
Using this notation, the optimal update can be obtained by applying a shrinkage operation. More specifically, when \( b_j = 0 \), we have the update given by
\[
x_j \leftarrow S_{\frac{\theta_1}{\theta_2}} \left( -\frac{D_1}{D_2} - T - \frac{\beta}{2D_2} \right) + T \tag{53}
\]
where the shrinkage operator is defined as
\[
S_{\lambda}(z) = \text{sign}(z) \max\{|z| - \lambda, 0\} , \tag{54}
\]
and when \( b_j = 1 \), we have
\[
x_j \leftarrow S_{\frac{\theta_1}{\theta_2}} \left( \frac{D_1}{D_2} - T + \frac{\beta}{2D_2} \right) + T \tag{55}
\]
Having obtained the optimal \( x_j \), we then re-allocate the projection using the projection update equation given by
\[
\begin{bmatrix} p_{L,i} \\ p_{H,i} \end{bmatrix} \leftarrow \begin{bmatrix} p_{L,i} \\ p_{H,i} \end{bmatrix} + \begin{bmatrix} A_{i,j}(1 - b_j) \\ A_{i,j} \end{bmatrix} (x_j - \bar{x}_j) . \tag{56}
\]
This will finish the ICD update of one specific pixel. Figure 2 summarizes the pseudocode of the ICD update for a specific pixel.
algorithm. This requires us to solve the 1D optimization of a function (40). We use the iterative conditional mode (ICM) for the configuration of $x$ initial reconstruction $\theta$ parameters according to (53) or (55) depending on the value of $b$. Finally, we update the projection vector using (56) and return the pixel update.

The actual implementation is to evaluate the 1D objective function (58) for $x$ and the material segmentation label $b$. Before and after the ICM update, the projection update is given by

$$ p_{L,i} \leftarrow p_{L,i} + A_{i,j}(x_j - \tilde{x}_j) $$

and

$$ p_{H,i} \leftarrow p_{H,i} + A_{i,j}(x_j - \tilde{x}_j) $$

The actual implementation is to evaluate the 1D objective function (58) for $x$ and the material segmentation $b$. If we let $\hat{x}_j$ and $b_j$ be the labels of the pixel before and after the ICM update, the projection update is given by

$$ p_{L,i} \leftarrow p_{L,i} + \hat{x}_j - b_j, \quad p_{H,i} \leftarrow p_{H,i} + \hat{x}_j - b_j $$

Figure 3 shows the pseudocode of the overall MBIR-BHC algorithm for simultaneous image reconstruction and beam hardening correction, which alternates over the optimization of the polynomial coefficients $\gamma$, the image $x$ and the material segmentation label $b$.

C. Optimization Over the Material Segmentation Mask

The segmentation vector $b$ is initialized by thresholding the initial reconstruction $x^{(\text{init})}$ at the beginning of the algorithm, given by

$$ b_j = \begin{cases} 0 & \text{if } x_j^{(\text{init})} \leq T \\ 1 & \text{otherwise} \end{cases} \quad (57) $$

During the alternating optimization, we fix $x$ and $\gamma$, and find the configuration of $b$ which minimizes the overall objective function (40). We use the iterative conditional mode (ICM) algorithm. This requires us to solve the 1D optimization of a particular segmentation label as follows

$$ b_j = \arg \min_{b_j \in \{0, 1\}} \left\{ \frac{1}{2} \sum_{i=1}^{M} w_i (y_i - h(p_i))^2 + \sum_{k \in \partial_j} \eta_{j,k}(1 - \delta(t - b_k)) \right\} + \beta((x_j - T)_+ (1 - t) + (T - x_j)_+ t) \quad (58) $$

The actual implementation is to evaluate the 1D objective function (58) for $t = 0$ or $1$ and to choose the optimal configuration of $t$ which gives the lower cost. After the label is updated, we adjust the projection according to this optimal configuration. If we let $\hat{b}_j$ and $b_j$ be the labels of the pixel before and after the ICM update, the projection update is given by

$$ p_{L,i} \leftarrow p_{L,i} + \left[ \hat{b}_j - b_j \right] A_{i,j} x_j \quad (59) $$

$\left[ \begin{array}{c} p_{L,i} \\ p_{H,i} \end{array} \right] \leftarrow \left[ \begin{array}{c} p_{L,i} \\ p_{H,i} \end{array} \right] + \left[ \begin{array}{c} \hat{b}_j - b_j \\ \hat{b}_j - b_j \end{array} \right] A_{i,j} x_j \quad (59) $$

Figure 3 shows the pseudocode of the overall MBIR-BHC algorithm for simultaneous image reconstruction and beam hardening correction, which alternates over the optimization of the polynomial coefficients $\gamma$, the image $x$ and the material segmentation label mask $b$.  

IV. RESULTS

In the following section, we evaluate MBIR with beam hardening correction (MBIR-BHC) using both simulated and real data sets, and we compare it to FBP, generic MBIR with a mono-energetic X-ray model (MBIR-mono), and the Empirical Beam Hardening Correction (EBHC) method [31]. The cost function for MBIR-mono is given by

$$ x \leftarrow \text{from FBP or generic MBIR} \quad \hat{p}_i \leftarrow \text{initial segment } x \text{ using (57)} \quad \text{for } i = 1 \text{ to } M \text{ do} \quad p_i \leftarrow \text{calculate using (20) and (21)} \quad \text{end for} \quad \text{repeat} \quad \gamma \leftarrow \text{solution of (41)} \quad d^{(i)} \leftarrow \text{calculate using (44) and (45)} \quad \text{end for} \quad \text{for } j = 1 \text{ to } N \text{ do} \quad \hat{x}_j \leftarrow \text{calculate using ICDUpdate in Figure 2} \quad \text{for } j = 1 \text{ to } N \text{ do} \quad b_j \leftarrow \text{solution of (58)} \quad p_i \leftarrow \text{updated using (59)} \quad \text{end for} \quad \text{for } i = 1 \text{ to } M \text{ do} \quad \hat{p}_i \leftarrow p_i \quad \text{end for} \quad \text{until convergence} $$

and we use the q-GGMRF potential function (37) for $\rho$. Both MBIR-mono and MBIR-BHC use the FBP reconstruction as an initial condition for optimization. Also, the segmentation, $b_j$, is initialized with a thresholded version of the FBP image, and we use a $3 \times 3$ neighborhood with coefficients $\alpha_{j,k}$ and $\eta_{j,k}$ selected so that

$$ \alpha_{j,k} = \alpha g_{j,k}, \quad \eta_{j,k} = \eta g_{j,k}. \quad (61) $$

Here $\alpha$, $\eta$ are two scalars and $g_{j,k}$ are the relative weights for different neighboring pixels, given by the following 2D array of values,

<table>
<thead>
<tr>
<th>0.11</th>
<th>0.14</th>
<th>0.11</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14</td>
<td>0</td>
<td>0.14</td>
</tr>
<tr>
<td>0.11</td>
<td>0.14</td>
<td>0.11</td>
</tr>
</tbody>
</table>
TABLE III
THE REGULARIZATION PARAMETER SETTING OF MBIR-BHC FOR DIFFERENT EXPERIMENTS

<table>
<thead>
<tr>
<th>Experiment</th>
<th>α</th>
<th>β</th>
<th>η</th>
</tr>
</thead>
<tbody>
<tr>
<td>two-material phantom</td>
<td>0.382</td>
<td>0.024</td>
<td>0.024</td>
</tr>
<tr>
<td>multi-material phantom (noiseless)</td>
<td>0.514</td>
<td>0.033</td>
<td>0.033</td>
</tr>
<tr>
<td>multi-material phantom (noisy)</td>
<td>0.013</td>
<td>0.033</td>
<td>0.033</td>
</tr>
<tr>
<td>modified NCAT phantom</td>
<td>0.630</td>
<td>0.040</td>
<td>0.040</td>
</tr>
<tr>
<td>real baggage scan</td>
<td>0.607</td>
<td>0.039</td>
<td>0.039</td>
</tr>
</tbody>
</table>

Fig. 4. Normalized energy spectrum of the X-ray source used for simulation where the center cell represents the pixel being considered. In Table III, we list the parameters α, β and η that were used for each experiment. For the EBHC method, the basic images are reconstructed using FBP. Unless otherwise stated, all MBIR-BHC results use a 2-nd order model with p = 2.

A. Simulation Results

In this section, we study the performance of different methods on various phantoms using the computer-simulated parallel-beam transmission polychromatic X-ray projections. The X-ray source spectrum we use is modelled using SPEC78 software from IPEM Report 78 [44] (tube voltage 95 kV, incident mean 56.4093 keV, std 14.2177 keV), and its normalized energy spectrum is plotted in Figure 4. Furthermore, in all the following simulation studies, the projections are pre-processed with respect to water using the standard polynomial fitting technique described in [3]. The resulting pre-corrected projection will be used as the input for the different methods.

1) The Two-Material Disk Phantom: The first phantom we study is a two-material phantom, made of a water disk with two aluminium insertions, as shown in Figure 5(a). The radius of the water disk and the aluminium insertions are 90 mm and 10 mm, respectively. The parallel-beam projection sinogram has 1024 detectors with 0.24 mm spacing and 720 projection angles over 180 degrees. We do not simulate the noise and scatter effects. All the reconstructed images are 512×512 over the 250 mm FOV. For MBIR-BHC, both the 2-nd and 3-rd order polynomial model are used and the segmentation threshold T is 800 HU. The reconstruction results using different methods are presented in Figure 5(b)-(i). Both FBP and MBIR-mono reconstructions contain streak artifacts due to the aluminium insertions. The EBHC method is able to partially suppress the streaks. However, the dark band connecting two insertions is still noticeable. The MBIR-BHC reduces the streak artifacts more effectively. However, in this case, the 3-rd order MBIR-BHC model seems to provide little benefit relative to the 2-nd order MBIR-BHC model.

TABLE IV
THE MEAN INTENSITY (HU) OF THE WATER REGION IN FIGURE 5

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Ground truth</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>FBP</td>
<td>-5.8</td>
<td></td>
</tr>
<tr>
<td>MBIR-mono</td>
<td>-7.5</td>
<td></td>
</tr>
<tr>
<td>EBHC</td>
<td>-44.2</td>
<td></td>
</tr>
<tr>
<td>2-nd order MBIR-BHC</td>
<td>-7.2</td>
<td></td>
</tr>
<tr>
<td>3-rd order MBIR-BHC</td>
<td>-7.3</td>
<td></td>
</tr>
</tbody>
</table>

TABLE V
QUANTITATIVE COMPARISON OF THE MODELING ERROR

<table>
<thead>
<tr>
<th>Experiment</th>
<th>MBIR-mono absolute mean</th>
<th>2-nd MBIR-BHC variance</th>
<th>3-rd MBIR-BHC variance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1187</td>
<td>0.1172</td>
<td>0.1177</td>
</tr>
<tr>
<td></td>
<td>0.0476</td>
<td>0.0436</td>
<td>0.0448</td>
</tr>
</tbody>
</table>

where the center cell represents the pixel being considered. In Table III, we list the parameters α, β and η that were used for each experiment. For the EBHC method, the basic images are reconstructed using FBP. Unless otherwise stated, all MBIR-BHC results use a 2-nd order model with p = 2.

A. Simulation Results

In this section, we study the performance of different methods on various phantoms using the computer-simulated parallel-beam transmission polychromatic X-ray projections. The X-ray source spectrum we use is modelled using SPEC78 software from IPEM Report 78 [44] (tube voltage 95 kV, incident mean 56.4093 keV, std 14.2177 keV), and its normalized energy spectrum is plotted in Figure 4. Furthermore, in all the following simulation studies, the projections are pre-processed with respect to water using the standard polynomial fitting technique described in [3]. The resulting pre-corrected projection will be used as the input for the different methods.

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2) The Multi-Material Disk Phantom: We continue the simulation study using a multi-material phantom, which consists of a water disk with radius of 90 mm, with several insertions of radius 10 mm, as shown in Figure 7(a). The chemical composition of the numbered objects are listed in Table VI. We first simulate a parallel-beam projection sinogram with 1024 detectors of 0.24 mm and 720 projection angles over...
Fig. 5. Comparison of FBP, MBIR-mono, EBHC and 2-nd and 3-rd order MBIR-BHC reconstructed images. Top row: the reconstructed images. Bottom row: the difference images. The display window for the reconstructed images is [-400 400] HU. The display window for the difference images is [-200 200] HU. Notice that the MBIR-BHC algorithm reduces visible beam-hardening artifacts in the reconstruction of simulated data.

Fig. 6. Comparison of the modeling error for the three forward models of MBIR-mono, MBIR-BHC with a 2-nd order model, and MBIR-BHC with a 3-rd order model. Notice that MBIR-BHC with the second order model produces the smallest modeling error.

TABLE VI
CHEMICAL COMPOSITION OF VARIOUS MATERIALS USED IN FIGURE 7(A)

<table>
<thead>
<tr>
<th>cylinder #</th>
<th>material</th>
<th>density (g/cc)</th>
<th>chemical formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>polypropylene</td>
<td>0.90</td>
<td>C₃H₆</td>
</tr>
<tr>
<td>2</td>
<td>polystryrene</td>
<td>1.05</td>
<td>C₈H₈O₂</td>
</tr>
<tr>
<td>3</td>
<td>acrylic</td>
<td>1.2</td>
<td>C₅H₈O₂</td>
</tr>
<tr>
<td>4</td>
<td>ULTEM</td>
<td>1.32</td>
<td>C₇H₂₄N₂O₆</td>
</tr>
<tr>
<td>5</td>
<td>ETFE</td>
<td>1.7</td>
<td>C₂H₄ + C₂F₄</td>
</tr>
<tr>
<td>8</td>
<td>PVC</td>
<td>1.4</td>
<td>C₂H₃Cl</td>
</tr>
</tbody>
</table>

We do not simulate the noise and scatter effects. All the reconstructed images are 512×512 over the 250 mm FOV. For MBIR-BHC, the 2-nd order polynomial model is used and the segmentation threshold T is 800 HU. The reconstruction results using different methods are presented in Figure 7. Notice that severe streaks through the high density objects are present in FBP and MBIR-mono reconstructions. In the EBHC reconstruction, streaks are suppressed but there are still noticeable artifacts remaining. In contrast, MBIR-BHC significantly reduces the streak artifacts. A corresponding improvement is also observed in the difference images. Figure 8 shows a comparison of the mean reconstructed values of the objects # 1-5 as compared to the theoretically correct values obtained from equation (6). The three algorithms of FBP, MBIR-mono and MBIR-BHC all produce attenuation coefficients with approximately equal accuracy, while EBHC introduces a bias in the mean reconstructed values.

Using the same simulated data from the multi-material phantom, we further investigate the effect of the order of the polynomial model in MBIR-BHC. Table VII lists the values of the polynomial coefficients estimated by the 2-nd and 3-rd order MBIR-BHC. Figure 9 compares the result of 2-nd and 3-rd order models for MBIR-BHC. By visual comparison, the 3-rd order MBIR-BHC reconstruction is slightly better than the 2-nd order result and the subtle improvement can...
Fig. 7. Comparison of FBP, MBIR-mono, EBHC and MBIR-BHC reconstructed images. Top row: the reconstructed images. Bottom row: the difference images. The display window for the reconstructed images is [-200 200] HU. The display window for the difference images is [-200 200] HU. Notice that the MBIR-BHC algorithm reduces visible beam-hardening artifacts in the reconstruction of simulated data.

(a) ground truth  (b) FBP  (c) MBIR-mono  (d) EBHC  (e) MBIR-BHC

(f) (b) - (a)  (g) (c) - (a)  (h) (d) - (a)  (i) (e) - (a)

Fig. 8. Comparison of the reconstruction accuracy of FBP, MBIR-mono, EBHC and MBIR-BHC. Ground truth values are calculated using (6). Notice that the MBIR-BHC reconstruction algorithm produces relatively low bias in the reconstructed density.

(a) 2-nd order MBIR-BHC  (b) 3-rd order MBIR-BHC  (c) (b) - (a)

Fig. 9. Comparison of 2-nd order MBIR-BHC and 3-rd order MBIR-BHC. The display window for the reconstructed images is [-200 200] HU. The display window for the difference image is [-50 50] HU. Notice that the 2-nd order model produces good results with lower complexity than the 3-rd order model.

be also noticed in the difference image. However, increasing parameters in higher order models may lead to the over-fitting of the projection data and also requires more computation. In practice, we have found that the 2-nd order MBIR-BHC model is sufficient to provide good results.

Figure 10 illustrates the robustness of the MBIR-BHC algorithm to sensor noise using the multi-material disk phantom. Our simulated data uses independent additive Gaussian noise with inverse variances given by (35) using $\lambda_{0,i} = 20000$ for all $i = 1, \cdots, M$ and $\sigma_e^2 = 16$. For the purposes of this particular comparison, we adjust the regularization in the MBIR-mono and MBIR-BHC to approximately match the noise variance of FBP. In practice, this means that the MBIR reconstructions are under-regularized since generally speaking MBIR can produce the same resolution as FBP at lower noise levels. Notice that both the FBP and MBIR-mono reconstructions contain streaks, while the MBIR-BHC effectively removes the streak artifacts even in the presence of noise.

3) The Modified NCAT Phantom: The third phantom we investigate is based on the NCAT phantom [45]. It has a FOV of 320 mm, and we manually inserted several regions of high density to simulate the presence of different materials.

![Table VII](image)

<table>
<thead>
<tr>
<th>model</th>
<th>$\tilde{\gamma}_{1,1}$</th>
<th>$\tilde{\gamma}_{1,2}$</th>
<th>$\tilde{\gamma}_{2,1}$</th>
<th>$\tilde{\gamma}_{1,2}$</th>
<th>$\tilde{\gamma}_{0,4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2-nd order</td>
<td>0.00945</td>
<td>-0.03231</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-nd order</td>
<td>0.03923</td>
<td>-0.03038</td>
<td>-0.00731</td>
<td>-0.00813</td>
<td>0.00511</td>
</tr>
</tbody>
</table>

Fig. 10. Comparison of FBP, MBIR-mono and MBIR-BHC reconstructions on noisy sinogram. The display window for the reconstructed images is [-200 200] HU. Notice that the MBIR-BHC algorithm robustly removes beam-hardening artifacts in the presence of simulated measurement noise.
and low density materials, such as soft tissue, bone, blood and metals, indicated in Figure 11. The densities and the mass attenuation coefficients of the materials used in the phantoms are obtained from the NIST XCOM database [46] and the chemical compositions of these materials are listed in Table VIII according to [47]. The simulated parallel-beam geometry has 1400 detectors with 0.23 mm spacing and 720 projection angles over 180 degrees. We do not simulate the noise and scatter effects. All the reconstructed images are $512 \times 512$. For MBIR-BHC, the 2-nd order polynomial model is used and the segmentation threshold $T$ is 800 HU. The reconstruction results are shown in Figure 13.

Due to the presence of high density metal insertions, the FBP reconstruction exhibits several streaking artifacts crossing through the image, which degrades the overall image quality. While the generic MBIR-mono reconstruction generally improves the image quality and smooths out several degraded regions, it fails to eliminate the streaks connecting the high density regions, such as bone and metal. The EBHC method is not as effective on this data set, probably due to the presence of multiple high density objects. By comparison, the MBIR-BHC method dramatically reduces most of the streaking artifacts, while producing a better rendering in uniform regions and preserving the shape of edges as well. The difference between the reconstruction and the ground truth also demonstrates the effectiveness of MBIR-BHC in removing streaks. We also plot the mean pixel values of different regions in the phantom versus their theoretical values in Figure 12. In the low attenuation region, FBP, MBIR-mono and MBIR-BHC give roughly equal accuracy estimates, while EBHC tends to under estimate the pixel value. In the high attenuation regions, all the methods produce lower accuracy estimates of density, but the MBIR-BHC method produces the most accurate results.

### B. Real Scan Data Results

In this section, we apply different methods to a real X-ray CT scan dataset taken of actual baggage with high and low density objects. The dataset is acquired from the Imatron C300
CT scanner, provided by the ALERT DHS center, Northeastern University, USA. The parallel-beam sinogram has 1024 detectors with 0.46 mm spacing and 720 projection angles over 180 degrees, rebinned from a fan-beam scan. The projection data is water pre-corrected. The reconstructed images have a FOV of 475 mm with the resolution of 512×512. We use the 2-nd order MBIR-BHC and choose the segmentation threshold $T$ to be 800 HU.

We first investigate the convergence behavior of the MBIR-BHC algorithm. Figure 14(a) plots the objective function after each iteration of optimization over $x$, $b$ and $\gamma$. As we have described in Section III, the objective function (40) is approximated by the quadratic approximation (43) using the second order Taylor series expansion with the expansion point iteratively refined during the reconstruction. While we point out that this approximation is not guaranteed to be an upper bound of the original objective function, which would ensure the convergence, we see from Figure 14(a) that in practice the objective function decreases, which suggests that the proposed strategy is effective for this high order non-linear optimization problem, and yields convergent results empirically. In Figure 14(b), we plot the change of the objective function relative to the converged value on a log scale, given by

$$\log \left( \frac{\text{cost}(t) - \text{cost}(\infty)}{\text{cost}(\infty)} \right)$$  \hspace{1cm} (64)

where $\text{cost}(\infty)$ is the converged value of the objective function. Note that the change is also monotonic decreasing. Figure 14(c) plots the average pixel change after each iteration in log scale. and Figure 14(d) plots the coefficients of the correction polynomial after each iteration. As shown in the figures, the average pixel change decreases to zero and the coefficients of the polynomial converge to a stable estimate after a few iterations.

Figure 15 shows the reconstruction results of this baggage scan dataset using different methods. By visual comparison to FBP and generic MBIR-mono, the MBIR-BHC method reduces the streaking and blooming artifacts significantly and produces reconstructions with higher resolution. This can be clearly observed in the zoomed-in region in Figure 16. The metal baggage handle causes a severe blooming artifact in the FBP image, and affects the nearby low density uniform regions as well. While the generic MBIR-mono algorithm recovers a few structures with better detail, it does not effectively address the artifacts due to the metal and many streaks remain on the final reconstructed image. On the other hand, the MBIR-BHC algorithm was able to produce more accurate and clearer images. The metal baggage handle causes a severe blooming artifact in the FBP image, and affects the nearby low density uniform regions. Under this assumption, the proposed method simultaneously estimates the reconstruction, the material segmentation mask, and the joint polynomial correction function of different material projections. Two separated projections are computed implicitly based on the low and high density materials during the iterative process. Therefore, no system information is needed and the correction is adapted to the dataset being used automatically. The experimental results on both the simulated and real dataset demonstrated the efficiency of the proposed algorithm in reducing several artifacts in the reconstructed image, such as streaking.

As seen from the results, MBIR-BHC has improved the overall image quality and significantly reduced the streaking artifacts due to beam hardening of the high density materials. Nonetheless, MBIR-BHC does not remove all beam-hardening artifacts. The remaining artifacts may be due to various reasons, including inaccurate modeling. For real scan data, other physical effects, such as scattering, may also influence the results. While Figure 6 indicates that the 2-nd order MBIR-BHC forward model is more accurate, the 3-rd order model shows little further improvement. This suggests that it may be possible to formulate different high-order models with less modeling error.

V. CONCLUSION

In this paper, we have presented a model-based iterative method for simultaneous image reconstruction and beam hardening correction (MBIR-BHC) which does not require any prior knowledge of the nonlinear characteristics of the system. The main idea is to jointly estimate the reconstruction and the beam hardening correction polynomial using an alternating optimization approach. The method is based on the assumption that distinct materials can be separated according to their densities. Under this assumption, the proposed method simultaneously estimates the reconstruction, the material segmentation mask, and the joint polynomial correction function of different material projections. Two separated projections are computed implicitly based on the low and high density materials during the iterative process. Therefore, no system information is needed and the correction is adapted to the dataset being used automatically. The experimental results on both the simulated and real dataset demonstrated the efficiency of the proposed algorithm in reducing several artifacts in the reconstructed image, such as streaking.
**APPENDIX A**

**DERIVATIONS OF THE RELATIONSHIPS IN SECTION II**

In this appendix, we derive the equations (14), (24)-(25), and (29)-(31) in Section II.

To see (14), we differentiate (11) and obtain

\[
\left. \frac{d}{dp_i} f_M(p_i) \right|_{p_i=0} = \frac{\int_{\mathbb{R}} \tilde{S}(E) e^{-r_M(E)p_i} dE}{\int_{\mathbb{R}} S(E) e^{-r_M(E)p_i} dE} \bigg|_{p_i=0} = \int_{\mathbb{R}} \tilde{S}(E) r_M(E) dE = 1
\]

where the last equality is the result of (9).

To see (24), we differentiate (19) and obtain

\[
\gamma_{1,0} = \left. \frac{\partial}{\partial p_{L,i}} h(p_{L,i}, 0) \right|_{p_{L,i}=0} = -\left. \frac{\partial}{\partial p_{L,i}} \log \left( \int_{\mathbb{R}} \tilde{S}(E) e^{-r_L(E)p_{L,i}} dE \right) \right|_{p_{L,i}=0} = \int_{\mathbb{R}} \tilde{S}(E) r_L(E) dE = 1
\]

and (25) can be derived in the similar manner, changing the differentiation with respect to \(p_{H,j}\).

For (29), plugging \(p_{L,i} = p_{H,j} = 0\) into (27), we obtain

\[
\tilde{\gamma}_{0,0} = f_M^{-1} \left( -\log \left( \int_{\mathbb{R}} \tilde{S}(E) e^0 dE \right) \right) = f^{-1}(0) = 0.
\]

Finally, to see (30), we apply the chain rule using (15) and (24) and obtain

\[
\gamma_{1,0} = \left. \frac{\partial}{\partial p_{L,i}} h(p_{L,i}, 0) \right|_{p_{L,i}=0} = \left. \frac{d}{dh(p_{L,i}, 0)} h(p_{L,i}, 0) \right|_{p_{L,i}=0} \frac{\partial}{\partial p_{L,i}} h(p_{L,i}, 0) \bigg|_{p_{L,i}=0} = \int_{\mathbb{R}} \tilde{S}(E) r_L(E) dE = 1
\]

and (31) can be derived in the similar manner, changing the differentiation with respect to \(p_{H,j}\).

**APPENDIX B**

**DERIVATION OF THE ICD UPDATE EQUATIONS**

In this appendix, we derive the ICD pixel update in detail. We rewrite (40) as a function of the pixel \(x_j\), drop the terms which are independent of \(x_j\), and obtain the 1D optimization problem over \(x_j\) given by

\[
x_j = \arg \min_{u \geq 0} \left\{ \theta_1(u - \tilde{x}_j) + \frac{1}{2} \theta_2(u - \tilde{x}_j)^2 + \beta ((u - T)_+ (1 - b_j) + (T - u)_+ b_j) + \sum_{k \in \partial_j} \alpha_{j,k} \rho(u - x_k) \right\}
\]

where \(\tilde{x}_j\) is the previous pixel value and \(\theta_1\) and \(\theta_2\) are given by (49) and (50). Following the similar strategy in [35], we simplify the optimization by replacing the function \(\rho\) with a quadratic substitute functional \(q\) given by

\[
q(u - x_k; \tilde{x}_j - x_k) = a_{j,k} (u - x_k)^2 + C_{j,k}
\]

where \(a_{j,k}\) and \(C_{j,k}\) are given by

\[
a_{j,k} = \rho'((\tilde{x}_j - x_k) / 2(\tilde{x}_j - x_k)) \quad \text{and} \quad C_{j,k} = \rho((\tilde{x}_j - x_k)) - a_{j,k} (\tilde{x}_j - x_k)^2.
\]

Fig. 13. Comparison of FBP, the generic MBIR with mono-energetic X-ray model (MBIR-mono), and MBIR-BHC reconstructed images. First row: the reconstructed images. Second row: the difference images. The display window for the reconstructed images is [-200 400] HU. The display window for the difference images is [-200 200] HU. Notice that the MBIR-BHC algorithm reduces visible beam-hardening artifacts in the reconstruction of the simulated human phantom.
It can be easily shown that this function $q$ is an upper bound of $\rho$, satisfying the two constraints given by

\begin{align}
q(u - x_k; \tilde{x}_j - x_k) &\geq \rho(u - x_k) , \\
q(\tilde{x}_j - x_k; \tilde{x}_j - x_k) &= \rho(\tilde{x}_j - x_k) .
\end{align}

Therefore, such substitution will not alter the optimal solution when the algorithm converges. Substituting (69) into (68) and dropping the terms which do not depend on $u$, we obtain the 1D optimization (47) with $\alpha'_{j,k}$ given by (48).

To derive the optimal solution to (47), we separate the cases when $b_j = 0$ or $1$. We first suppose $b_j = 0$ and zero the derivative of (47) to obtain the normal equation given by

\begin{equation}
\theta_1 - \theta_2 \tilde{x}_j - 2 \sum_{k \in \partial j} \alpha'_{j,k} x_k + \left( \theta_2 + 2 \sum_{k \in \partial j} \alpha'_{j,k} \right) u = -\beta \text{step}(u - T) \\
D_1 + D_2 u = -\beta \text{step}(u - T)
\end{equation}

where step$(\cdot)$ is the unit step function, and $D_1$ and $D_2$ are given by (51) and (52). The solution to (74) can be obtained using the shrinkage operator in solving LASSO and $l_1$ regularized problems [48], and the closed form solution is given by (53). For the case when $b_j = 1$, we have the normal equation given by

\begin{equation}
D_1 + D_2 u = \beta \text{step}(T - u) .
\end{equation}
Fig. 16. Comparison of FBP, MBIR-mono, MBIR-BHC on the real X-ray CT data of a scanned baggage. Images are zoomed-in a region including metal objects. The display window is \([-1000, 1000]\) HU. Notice that the MBIR-BHC reconstructions have reduced streaking, more uniform homogeneous regions, and less blooming of metal in this real data set.

Fig. 17. Comparison of MBIR-mono and MBIR-BHC on the normalized error sinogram and the update equation in this case becomes (55). Figure 18 illustrates these two cases graphically.

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