# EFFICIENT ML ESTIMATION OF THE SHAPE PARAMETER FOR GENERALIZED GAUSSIAN MRF

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## ABSTRACT

A certain class of Markov Random Fields (MRF) known as generalized Gaussian MRFs (GGMRF) have been shown to yield good performance in modeling the *a priori* information in Bayesian image reconstruction and restoration problems. Though the ML estimate of temperature $T$ of a GGMRF has a closed form solution, the optimal estimation of the shape parameter $p$ is a difficult problem due to the intractable nature of the partition function. In this paper, we present a tractable scheme for ML estimation of $p$ by an off-line numerical computation of the log of the partition function. In image reconstruction or restoration problems, the image itself is not known. To address this problem, we use the EM algorithm to compute the estimates directly from the data. For efficient computation of the expectation step, we propose a fast simulation technique and a method to extrapolate the estimates when the simulations are terminated prematurely prior to convergence. Experimental results show that the proposed methods result in substantial savings in computation and superior quality images.

## 1. INTRODUCTION

Bayesian estimation methods have become popular in recent times for image reconstruction and restoration problems. Markov random fields (MRF) have proven useful in modeling the *a priori* information in these methods. The MRF model is equivalent to a Gibbs distribution and is often specified in terms of a potential function which assigns a cost to differences between neighboring pixels. The preponderance of the previous work has focused primarily on the quadratic choice for the potential function or Gaussian MRF. Although this particular choice has many analytical advantages, the edges in the reconstruction are blurred due to the excessive cost assigned to abrupt transitions. Many alternative potential functions have been proposed in the literature which help to alleviate this problem [1, 2, 3, 4]. In particular, the generalized Gaussian MRF (GGMRF) [4] uses a potential function similar to the log of the generalized Gaussian noise density found commonly in robust detection and estimation. It renders edges accurately without prior knowledge of their size, and it results in a convex optimization problem with a unique global minimum [4].

The GGMRF has two parameters associated with it - temperature $T$ and shape parameter $p$. Bouman and Sauer [5] have shown that the ML estimate of temperature $T$ has a very simple closed form for the GGMRF case. However, the ML estimation of the shape parameter $p$ remains a difficult problem due to the intractable nature of the partition function. To circumvent this difficulty, Pun and Jeffs [6] have suggested an alternate method to estimate the shape parameter by computing the kurtosis of differences between neighboring pixels. However, in this paper, we solve the optimal ML estimation problem by an off-line numerical computation of the log of the partition function.

In the context of image reconstruction or restoration problem, the difficulty of the ML estimation problem is compounded by the fact that the estimates depend on the unknown image. Geman and McClure [7] and later Bouman and Sauer [5] solved the incomplete data problem by using the EM algorithm. Since the expectation involved in the EM algorithm is intractable, a stochastic simulation method is used to generate samples of the unknown image from the posterior distribution. A commonly used simulation method, the Metropolis algorithm tends to suffer from slow convergence to the equilibrium distribution. In this paper, we derive a fast simulation technique to facilitate the expectation step. We also propose a method to extrapolate the estimates when the simulations are terminated prematurely prior to convergence. Experimental results show that the proposed methods result in substantial savings in computation and superior quality images.

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## 2. ML ESTIMATE OF $P$ FOR GGMRF

Let upper case letters denote random variables and let lower case letters denote realizations of random variables. Let $x$ be the image. The normalized log likelihood of $x$ modeled as a GGMRF is given as

$$\frac{\log \mathcal{P}(x)}{N} = \frac{-1}{NT} u_p(x) - \frac{1}{N} \log z_p(T) \tag{1}$$

where $z_p(T)$ is the partition function and $u_p(\cdot)$ is the energy function of the form

$$u_p(x) = \sum_{\{i,j\} \in \mathcal{N}} b_{i-j} |x_i - x_j|^p \tag{2}$$

where $\mathcal{N}$ is the set of all neighboring pixel pairs.

Using the normalized log likelihood of $x$, the joint ML estimation of $T$ and $p$ can be expressed as a two dimensional optimization problem

$$[\hat{T}, \hat{p}] = \arg \min_{[T,p]} \left\{ \frac{1}{NT} u_p(x) + \frac{1}{N} log z_p(T) \right\} \tag{3}$$

It was recently shown by Bouman and Sauer [5] that $z_p(T)$ can be expressed as

$$z_p(T) = T^{N/p} z_p(1). \tag{4}$$

They used the above result to obtain the ML estimate of $T$ as

$$\hat{T}(p, x) = \frac{p u_p(x)}{N} \tag{5}$$

Using (4) and (5), we can reduce (3) to a one dimensional optimization to obtain the ML estimate of $p$ as

$$\hat{p}(x) = \arg\min_p \left\{ \frac{\log \hat{T}(p,x)}{p} + \frac{1}{p} + f(p) \right\} \tag{6}$$

where

$$f(p) = \frac{\log z_p(1)}{N} = \frac{1}{N} \log \int_{x \in \mathbb{R}^{+N}} \exp\{-u_p(x)\}\, dx$$

Note that the $\hat{T}(p, x)$ is a sufficient statistic to determine the ML estimate of $p$.

Direct computation of $f(p)$ would require the numerical evaluation of an $N$ dimensional integral which is not feasible. Instead a more elegant method of computing $f(p)$ is through its derivative.

$$f'(p) = -E\left[ \frac{1}{N} \frac{d}{dp} u_p(X) \,|T = 1,\ p \right]$$

Note the normalization by $N$ of $f'(p)$ is essential for the function to be useful for any image size. The function $f'(p)$ is computed off-line prior to the estimation procedure. A second order spline is fitted to $f'(p)$ and integrated to obtain $f(p)$. Fig. 1 shows the log partition function and its derivative for a second order neighborhood using periodic boundary conditions.

The ML estimate of $p$ is obtained by computing (6) for a finely spaced set of values for $0.8 \leq p \leq 2$ and finding the minimum with respect to $p$ [1]. The lower limit of 0.8 was set arbitrarily considering that we are only interested in $p \geq 1$ ( since the GGMRF prior is non-convex for $p < 1$ and hence not desirable for MAP estimation).

In image reconstruction or restoration problems, $x$ is unknown. We use the EM algorithm in this case to compute the estimates. The EM algorithm results in an iterative procedure where the parameters are updated with each iteration $k$ in the following fashion

$$p^{k+1} = \arg\min_p \left\{ \log \bar{T}^k(p) + \frac{1}{p} + f(p) \right\} \tag{7}$$

$$T^{k+1} = \bar{T}^k(p^{k+1}) \tag{8}$$

where

$$\bar{T}^k(p) \triangleq \frac{p}{N} E\left[ u_p(X) \,\middle|\, Y = y,\ T^k,\ p^k \right]$$

Refer to [8] for the derivation. The required conditional expectation can be computed by generating samples from the posterior distribution of the cross-section $x$ given $y$ and $[T^k, p^k]$.

[1] We could also reduce this computation by employing a fast rooting method such as the half interval search to root the derivative of $\log \mathcal{P}(x)$ with respect to $p$.

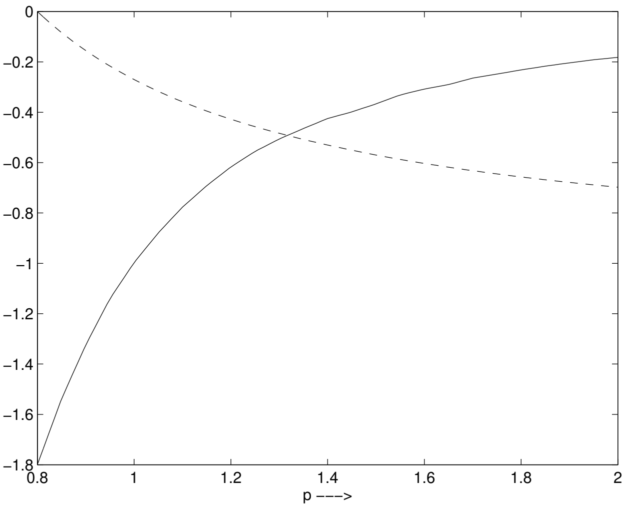


Figure 1: The solid line shows $f'(p)$ and the dashed line shows $f(p)$ for a second order neighborhood using periodic boundary conditions.

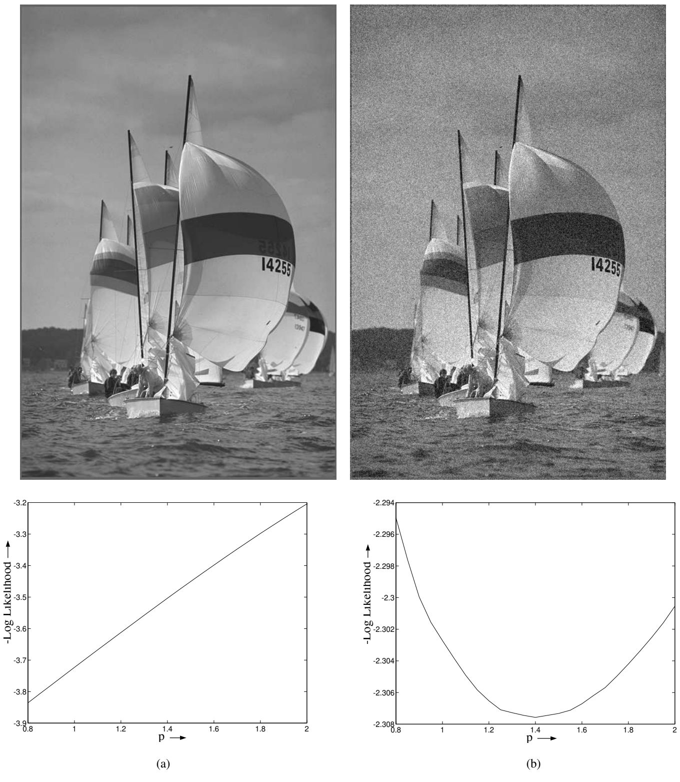


Figure 2: ML estimation of $p$ for natural images. The plot below each image shows the corresponding negative log likelihood as a function of $p$. The ML estimate is the value of $p$ that minimizes the plotted function.

## 3. FAST SIMULATION TECHNIQUE

The Metropolis algorithm is a commonly used simulation method for generating samples from a given distribution. However, this algorithm tends to suffer from slow convergence to the equilibrium distribution because the transition probability distribution is required to be symmetric.

Hastings [9] and Peskun [10] developed a generalization of the Metropolis algorithm which compensates for asymmetric transition probabilities through the proper choice of the associated acceptance probability. Green and Han [11] have argued that convergence is fastest if the transition probability is chosen to be close to that of the Gibbs sampler. This can be done by approximating each pixel's marginal distribution by a Gaussian distribution.

In the context of the emission or transmission tomography, a Gaussian transition probability is not always effective because of the positivity constraint. Therefore, we first compute the MAP update ($\mu$) at a pixel. If the $\mu$ is positive, then we use a truncated Gaussian transition probability with mean $\mu$ and appropriate variance. If the $\mu$ is negative, then we use a strictly positive exponential distribution with appropriate mean. Refer to [8] for the details of the algorithm.

When the simulations are terminated prematurely, we need to extrapolate the parameters to obtain estimates closer to the true value. Towards this end, we derive in [8] the gradient of the log likelihood with respect to $T$ and $p$ as

$$\frac{1}{N}\frac{\partial}{\partial T}\log P(y) = \frac{1}{NT^2}E\left[u_p(X)|\,Y=y,T,p\right] - \frac{1}{pT}$$

$$\frac{1}{N}\frac{\partial}{\partial p}\log P(y) = \frac{-1}{NT}E\left[\frac{d}{dp}u_p(X)\mid Y=y,T,p\right] + \frac{1}{p^2}\log T - f'(p)$$

Note that these gradients can be computed easily at the current value of the parameters when the EM updates are computed. We then do a least squares fit of the gradient(s) to a line (plane) using the past $n$ samples of the gradient(s) when we update $T$ ($p$ and $T$). The zero crossing of the fit then determines the extrapolated estimates of the parameters.

## 4. EXPERIMENTAL RESULTS

We computed the ML estimate of $p$ for a host of natural images. Fig. 2 shows the negative log likelihood with respect to $p$ of one such image. For the original image, the estimate hits the constraint of 0.8. In fact, this was the case for most of the images that we tried. Consequently natural images appear more Laplacian than Gaussian. The second image in Fig. 2 is obtained by adding Gaussian noise to the original image. In this case we observe that the ML estimate of $p$ rises to 1.4.

Fig. 4 shows the emission phantom used in our experiments. Poisson random variables were generated from 128 projections taken at 128 uniformly spaced angles to obtain the noisy projection data. The generalized Hamming filter was used in the Convolution Back Projection (CBP) algorithm to reconstruct the phantom at a 128 by 128 pixel resolution.

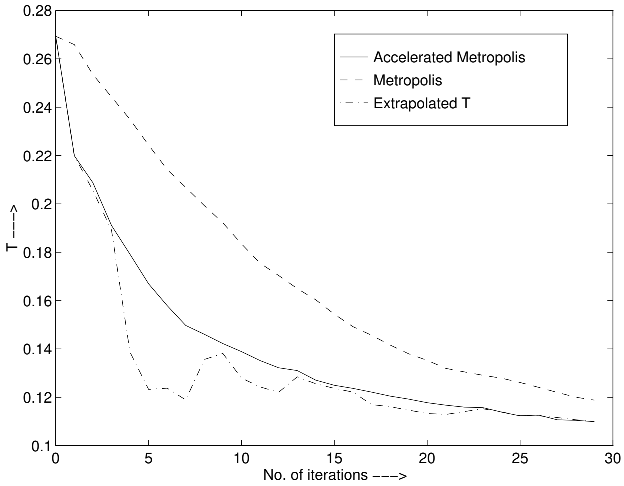


Figure 3: EM updates of $T$ using the Metropolis and the accelerated Metropolis algorithm for the emission phantom. $p = 1.1$ is assumed known. 5 samples from the past are used for the least squares fit to obtain the extrapolated estimate.

Fig. 3 compares the fast simulation method and the extrapolated estimates with the conventional Metropolis algorithm when $p$ is assumed known and $T$ is estimated using the EM algorithm. Note the extrapolated estimate is close to the true value after just 5 iterations. Fig. 4 shows the reconstructed phantom using the estimated value of $T$.

Fig. 5 shows the transmission phantom. The attenuation map is a 128 by 64 array of 4.5mm pixels; it represents a human thorax with linear attenuation coefficients 0.0165/mm, 0.0096/mm, and 0.0025/mm, for bone, soft tissue, and lungs respectively. Poisson random variables were generated from 192 projections taken at 256 uniformly spaced angles to obtain the noisy projection data. Fig. 5 also shows the CBP image and the MAP reconstruction using a GGMRF prior with $p = 1.1$ and the ML estimate of $T$.

## 5. REFERENCES


[1] A. Blake and A. Zisserman, *Visual Reconstruction.* Cambridge, Massachusetts: MIT Press, 1987.

[2] P. J. Green, "Bayesian reconstruction from emission tomography data using a modified EM algorithm," *IEEE Trans. on Medical Imaging*, vol. 9, no. 1, pp. 84–93, March 1990.

[3] R. Stevenson and E. Delp, "Fitting curves with discontinuities," *Proc. of the first international workshop on robust computer vision*, pp. 127–136, October 1-3 1990.

[4] C. A. Bouman and K. Sauer, "A generalized Gaussian image model for edge-preserving map estimation,"

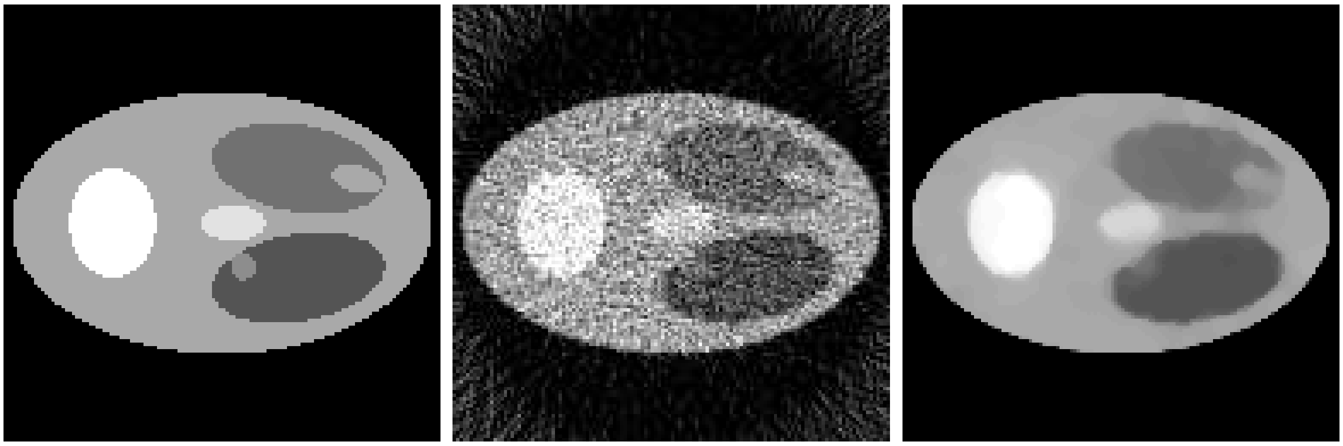

Figure 4: From left to right a) Emission phantom b) CBP image c) Reconstructed phantom with $p = 1.1$ assumed known and $T$ estimated using the EM algorithm.

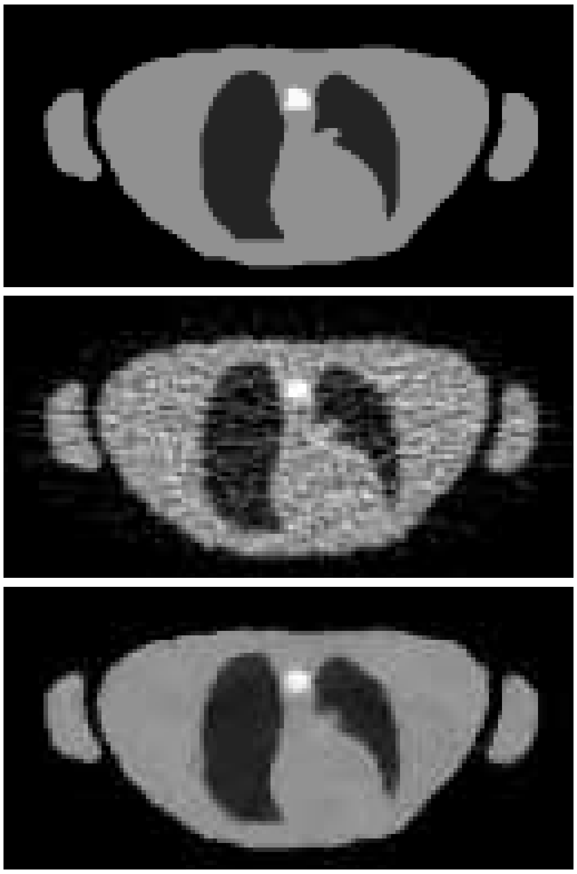

Figure 5: From top to bottom a) Transmission phantom b) CBP image c) Reconstructed phantom with $p = 1.1$ assumed known and $T$ estimated using the EM algorithm.

*IEEE Trans. on Image Processing*, vol. 2, pp. 296–310, July 1993.

[5] C. A. Bouman and K. Sauer, "Maximum likelihood scale estimation for a class of Markov random fields," *Proc. of IEEE Int'l Conf. on Acoust., Speech and Sig. Proc.*, vol. 5, April 19-22 1994, Adelaide, South Australia, pp. 537–540.

[6] W. Pun and B. Jeffs, "Shape parameter estimation for generalized Gaussian Markov random field models used in MAP image restoration," *29th Asilomar Conference on Signals, Systems, and Computers*, Oct. 29 - Nov. 1 1995.

[7] S. Geman and D. McClure, "Statistical methods for tomographic image reconstruction," *Bull. Int. Stat. Inst.*, vol. LII-4, pp. 5–21, 1987.

[8] S. S. Saquib, C. A. Bouman, and K. Sauer, "ML parameter estimation for Markov random fields, with applications to Bayesian tomography," Tech. Rep. TR-ECE 95-24, School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907, October 1995.

[9] W. K. Hastings, "Monte Carlo sampling methods using Markov chains and their applications," *Biometrika*, vol. 57, no. 1, pp. 97–109, 1970.

[10] P. H. Peskun, "Optimum Monte-Carlo sampling using Markov chains," *Biometrika*, vol. 60, no. 3, pp. 607–612, 1973.

[11] P. J. Green and X. liang Han, "Metropolis methods, Gaussian proposals and antithetic variables," in *Stochastic Models, Statistical methods, and Algorithms in Image Analysis* (P. Barone, A. Frigessi, and M. Piccioni, eds.), pp. 142–164, Berlin: Springer-Verlag, 1992.