EE 641 Midterm Exam

Spring 1998

Name: ______________________________ Starting time: ________________________

Ending time: ________________________

Instructions

The following is a take home exam.

- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- Answer questions precisely and completely. Credit will be subtracted off for vague answers.
- Each question is worth 20 points, so make sure that you get easy parts right before doing difficult parts.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and your EE600 and EE638 text books. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 743-0744 (8:00AM to 8:00PM); or office 494-0340, or send email at bouman@ecn.purdue.edu.

Good luck.

Problem 1.(20pt)

Let $\{X_n\}_{n=1}^N$ be a 1-D discrete state MRF with $X_n \in \{0, \cdots, M\}$ and Gibbs distribution

$$p(x) = \frac{1}{z} \exp\{-\sum_{n=2}^{N} v_n(x_n, x_{n-1})\}$$

where $v_n(\cdot, \cdot)$ is an arbitary potential function.

a) Calculate an expression for the probability mass function $p(x_n|x_i \; i \neq n)$ for $2 \leq n \leq N-1$.

b) Show that for any choice of $v_n(\cdot, \cdot)$, X_n is a Markov chain.

c) Find potential functions $v_n(j, i)$ so that $\{X_n\}_{n=1}^N$ is a homogeneous Markov chain with $P(X_1 = i) = \pi_i$ and transition probabilities $P_{i,j}$.

Problem 2.(20pt)

Consider the random variable X with density

$$p(x) = \begin{cases} \frac{1}{z} \exp\{-x^4\} & \text{for } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Propose a method based on the Hastings-Metropolis algorithm for generating samples of X and use a sampling distribution of $q(x, x') = f(x')$ for some density $f(x')$.

b) Derive the specific algorithm when $f(x)$ is exponentially distributed with mean μ.

c) Derive the specific algorithm when $f(x) = p(x)$. Explain your results.

Problem 3.(20pt)

Consider the 1-D Gaussian MRF $\{X_n\}_{n=0}^{N-1}$ with energy function

$$u(x) = \sum_{n=1}^{N-1} (x_n - x_{n-1})^2 + (x_0 - x_{N-1})^2 + \sum_{n=0}^{N-1} x_n^2$$

and Gibbs distribution $p(x) = \frac{1}{z} \exp\{-u(x)\}$.

a) Calculate the density $p(x_n|x_i \text{ for } i \neq n)$.

b) For $N = 5$, calculate the matrix B and the vector μ so that

$$u(x) = (x - \mu)^t B(x - \mu)$$

where $x = [x_0, \cdots, x_{N-1}]^t$.

c) Show that the vector $e_k = [1, \cos(2\pi k/N) \cdots, \cos(2\pi 2k/N), \cdots, \cos(2\pi nk/N)]^t$ is an eigenvector of the matrix B and compute its corresponding eigenvalue.

d) Use the result of part c) to compute an expression for

$$\lim_{N \to \infty} \frac{\log z}{N}$$

Problem 4.(20pt)

Let X be an N point MRF with a Gibbs distribution given by

$$p(x) = \frac{1}{z(\sigma)} \exp\left\{ - \sum_{\{s,r\}\in C} \frac{1}{p\sigma^p} |x_s - x_r|^p \right\}$$

Let Y be a second random field such that $Y = X + N$ where N is composed of i.i.d. $N(0,1)$ Gaussian random variables.

a) Show that $z(\sigma) = \sigma^N z(1)$.

b) Use the result of a) to derive a simple expression for the ML estimate of σ^p.

For the following parts assume that Y can be observed, but that X can not be directly observed.

c) Calculate the update equations for the estimation of σ^p using the EM algorithm. Express your result in terms of the probability density $p(x|y)$.

d) Propose a method for generating random samples from the distribution $p(x|y)$ by using the Metropolis algorithm.

e) Explain how the EM algorithm may be approximately implemented by using the results of parts c) and d). Be specific.

Problem 5.(20pt)

Let $\{X_n\}_{n=0}^{N-1}$ be a sequence of N i.i.d. binary random variables that take on values in the set $\{0, 1\}$ with probability $P(X_n = 0) = \pi$. Furthermore, let Y_n be continuous random variables that are conditionally independent given X_n with conditional density function $f(y_n|x_n) > 0$ for all $y_n \in I\!R$ and $x_n \in \{0, 1\}$.

a) Calculate the log marginal density $\log p(y)$.

b) Show that $\log p(y)$ is a strictly concave function of π.

c) Compute the function $Q(\pi', \pi)$ for EM algorithm.

d) Write the full EM algorithm update equations for $\pi^{(k+1)}$ in terms of $\pi^{(k)}$ and $f(y_n|x_n)$.