Midterm Solutions

1) a)

$$p(x_n \mid x_i \; i \neq n) = \frac{\exp\{-U_n(x_n, x_{n-1}) - U_{n+1}(x_{n+1}, x_n)\}}{\sum_{x_n} \exp\{-U_n(x_n, x_{n-1}) - U_{n+1}(x_{n+1}, x_n)\}}$$

b)

$$p(x_i \; i \geq n \mid x_j \; j < n) = \frac{p(x)}{p(x_j \; j < n)}$$

$$= \frac{\exp\left\{-\sum_{i=n}^{N} V_i(x_i, x_{i-1})\right\} \cancel{\exp\left\{-\sum_{i=0}^{n-1} V_i(x_i, x_{i-1})\right\}}}{\sum_{x_i \; i \geq n+1} \exp\left\{-\sum_{i=n}^{N} V_i(x_i, x_{i-1})\right\} \cancel{\exp\left\{-\sum_{i=0}^{n-1} V_i(x_i, x_{i-1})\right\}}}$$

not a function of $x_j \; j \leq n-2$

$$= p(x_i \; i \geq n \mid x_{n-1})$$

So

$$p(x_n \mid x_j \; j < n) = \sum_{x_k \; k \geq n+1} p(x_i \; i \geq n \mid x_j \; j < n)$$

$$= \sum_{x_k \; k \geq n+1} p(x_i \; i \geq n \mid x_{n-1})$$

$$= p(x_n \mid x_{n-1}) \qquad QED$$

e) $p(x) = \pi_{x_1} \prod_{n=2}^{N} P_{x_{n-1} x_n}$

$$\log p(x) = \log \pi_{x_1} + \sum_{n=1}^{N} \log P_{x_{n-1} x_n}$$

$$V_1(j,i) = -\log \pi_i - \log P_{ij}$$

$$N \geq n > 1 \quad V_n(j,i) = -\log P_{ij}$$

2) a)

$$\alpha(x, x') = \min\left\{1, \frac{p(x')}{p(x)} \frac{f(x)}{f(x')}\right\}$$

b) $$f(x) = \begin{cases} \frac{1}{\mu} e^{-\frac{1}{\mu} x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$\frac{p(x')}{p(x)} \frac{f(x)}{f(x')} = \frac{e^{-x'^4}}{e^{-x^4}} \frac{e^{-\frac{1}{\mu} x}}{e^{-\frac{1}{\mu} x'}}$$

$$= \exp\left\{ x^4 - x'^4 + \frac{1}{\mu}(x' - x) \right\}$$

$$\alpha(x, x') = \min\left\{1, \exp\left\{x^4 - x'^4 + \frac{1}{\mu}(x' - x)\right\}\right\}$$

c) $f(x) = p(x)$

$$\frac{p(x')}{p(x)} \frac{f(x)}{f(x')} = 1 \Rightarrow \alpha(x, x') = 1$$

always accept!

3) a)

$$p(x_n \mid x_i \; i \neq n) = \frac{1}{Z} \exp\left\{ -(x_n - x_{n-1})^2 - (x_n - x_{n+1})^2 - x_n^2 \right\}$$

$$(x_n - x_{n-1})^2 + (x_n - x_{n+1})^2 + x_n^2 = \frac{1}{2\sigma^2}(x_n - \mu)^2$$

$$\sigma^2 = 1/6$$

$$\mu = \frac{x_{n-1} + x_{n+1}}{3}$$

$$p(x_n \mid x_i \; i \neq n) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ \frac{-1}{2\sigma^2}(x_n - \mu)^2 \right\}$$

b) $\mu = 0$

$$B = \begin{bmatrix} 3 & -1 & 0 & 0 & -1 \\ -1 & 3 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -1 \\ -1 & 0 & 0 & -1 & 3 \end{bmatrix}$$

c) Let

$$\tilde{e}_k = e^{-j2\pi kn/N}$$

$$\sum_k B_{ik}\tilde{e}_k = 3e^{-j2\pi kn/N} - e^{-j2\pi k(n+1)/N} - e^{-j2\pi k(n-1)/N}$$

$$= e^{-j2\pi kn/N}(3 - 2\cos(2\pi k/N))$$

so $\quad B\tilde{e}_k = \tilde{e}_k\lambda_k$

$$\lambda_k = (3 - 2\cos(2\pi k/N))$$

Notice that $\quad e_k = \dfrac{\tilde{e}_k + \tilde{e}_k^*}{2}$

so

$$Be_k = e_k\lambda_k$$

$$\lambda_k = 3 - 2\cos(2\pi k/N)$$

d) Therefore

$$B = E \Lambda E^{+}$$

where $E = [e_0, \ldots, e_{N-1}]$

$$\Lambda = \mathrm{diag}\{\lambda_0, \ldots, \lambda_{N-1}\}$$

$$p(x) = \frac{1}{Z} \exp\{-x^{+} B x\}$$

$$Z = (2\pi)^{N/2} |B|^{-1/2}$$

$$= (2\pi)^{N/2} |\Lambda|^{1/2}$$

$$\frac{\log Z}{N} = \frac{1}{2}\log(2\pi) + \frac{1}{2N}\sum_{k=0}^{N-1} \log \lambda_k$$

$$= \frac{1}{2}\log(2\pi) + \frac{1}{2N}\sum_{k=0}^{N-1} \log\left(3 - 2\cos\left(2\pi\frac{k}{N}\right)\right)$$

$$\lim_{N\to\infty} \frac{\log Z}{N} = \frac{1}{2}\log(2\pi) + \frac{1}{2}\int_0^1 \log(3 - 2\cos(2\pi f))\,df$$

4) a) Let $u(x,\sigma) = \sum_{\{s,r\}\in C} \frac{1}{p\sigma^p} |x_s - x_r|^p$

$$= \sum_{\{s,r\}\in C} \frac{1}{p} \left| \frac{x_s - x_r}{\sigma} \right|^p$$

$$= u\left(\frac{x}{\sigma}, 1\right) \triangleq u\left(\frac{x}{\sigma}\right)$$

$$Z(\sigma) = \int_{\mathbb{R}^N} e^{-u\left(\frac{x}{\sigma}\right)} dx$$

$$= \int_{\mathbb{R}^N} e^{-u(x)} \sigma^N dx$$

$$= \sigma^N Z(1)$$

b) $\log p(x|\sigma) = -u(x/\sigma) - \log Z(\sigma)$

$$= -u(x/\sigma) - N \log \sigma - \log Z(1)$$

$$\frac{1}{N} \log p(x|\sigma) = -\frac{1}{N} u(x/\sigma) - \log \sigma - \frac{1}{N} \log Z(1)$$

$$\frac{d}{d\sigma} \frac{1}{N} \log p(x|\sigma) = \frac{d}{d\sigma}\left(-\frac{1}{N} u(x/\sigma)\right) - \frac{1}{\sigma}$$

notice that

$$u(x/\sigma) = \frac{1}{\sigma^p} u(x)$$

$$\frac{d}{d\sigma} \frac{1}{N} \log p(x|\sigma) = -\frac{1}{N} u(x) (-p) \sigma^{-p-1} - \sigma^{-1}$$

$$= 0 \Leftarrow \text{ML estimate}$$

$$\frac{p}{N} u(x)\, \sigma^{-p} - 1 = 0$$

$$\hat{\sigma}^{p} = \frac{p}{N} u(x)$$

$$\hat{\sigma}^{p} = \frac{1}{N} \sum_{\{s,r\}\in C} |x_s - x_r|^{p}$$

c) Define $\gamma = \sigma^{p}$

$$Q(\gamma, \gamma') = E\left[\log p(x|\gamma) \,/\, Y, \gamma'\right]$$

$$= E\left[-u(x/\sigma) - N\log\sigma - \log z(1) \,\middle|\, Y, \gamma'\right]$$

$$= E\left[-\frac{1}{\sigma} u(x) - N\log\sigma - \log z(1) \,\middle|\, Y, \gamma'\right]$$

$$= -\frac{1}{\sigma} \underbrace{E\left[u(x) / Y, \gamma'\right]}_{\bar{u}(x)} - N\log\sigma - \log z(1)$$

$$\gamma^{(k+1)} = \arg\max_{\gamma} Q(\gamma, \gamma^{(k)})$$

$$= E\left[\frac{1}{N} \sum_{\{s,r\}\in C} |x_s - x_r|^{p} \,\middle|\, Y, \gamma^{(k)}\right]$$

5)

a) $p(Y) = \prod_{n=0}^{N-1} p(y_n)$

$= \prod_{n=0}^{N-1} \left(p(y_n|0)\pi + p(y_n|1)(1-\pi) \right)$

$$\log p(Y) = \sum_{n=0}^{N-1} \log\left(p(y_n|0)\pi + p(y_n|1)(1-\pi) \right)$$

$$= \sum_{n=0}^{N-1} \log\left(\pi\left(p(y_n|0) + p(y_n|1)\right) + p(y_n|1) \right)$$

b) $$\frac{d^2}{d\pi^2} p(y) = \sum_{n=0}^{N-1} \frac{-1}{\{\pi(p(y_n|0)+p(y_n|1))+p(y_n|1)\}^2} \cdot (p(y_n|0)+p(y_n|1)) \cdot p(y_n|0)$$

$< 0 \Rightarrow$ strictly convex

c) $Q(\pi', \pi) = E\left[\log p(Y, X|\pi') \,|\, Y, \pi \right]$

$$\log p(y, x|\pi') = \sum_{n=0}^{N-1} \left\{ \log p(y_n|x_n) + (\log \pi')\,\delta_{x_n} + (\log(1-\pi'))\,\delta_{x_n - 1} \right\}$$

$$Q(\pi', \pi) = \sum_{n=0}^{N-1} (\log \pi')\, E\left[\delta_{x_n} | Y, \pi\right] + \sum_{n=0}^{N-1} \log(1-\pi')\, E\left[\delta_{x_n-1} | Y, \pi\right]$$

$+ \text{CONST}$

not a function of π

$$E[\delta_{x_n} \mid Y, \pi] = p(X_n = 0 \mid Y_n, \pi)$$

$$= \frac{p(y_n \mid 0)\,\pi}{p(y_n \mid 0)\,\pi + p(y_n \mid 1)(1-\pi)}$$

$$E[\delta_{x_n - 1} \mid Y, \pi] = \frac{p(y_n \mid 1)(1-\pi)}{p(y_n \mid 0)\,\pi + p(y_n \mid 1)(1-\pi)}$$

$$Q(\pi', \pi) =$$

$$\sum_{n=0}^{N-1} \log \pi' \frac{p(y_n \mid 0)\,\pi}{p(y_n \mid 0)\,\pi + p(y_n \mid 1)(1-\pi)}$$

$$+ \sum_{n=0}^{N-1} \log(1-\pi') \frac{p(y_n \mid 1)(1-\pi)}{p(y_n \mid 0)\,\pi + p(y_n \mid 1)(1-\pi)}$$

$$+ \text{CONST}$$

d)

$$\pi^{(k+1)} = \arg\max_{\pi'} Q(\pi', \pi^{(k)})$$

$$\frac{1}{N} \sum_{n=0}^{N-1} \frac{p(y_n \mid 0)\,\pi^{(k)}}{p(y_n \mid 0)\,\pi^{(k)} + p(y_n \mid 1)(1-\pi^{(k)})}$$