EE 641 Midterm Exam

Spring 1996

Name: ______________________ Starting time: ______________________

Ending time: ______________________

Instructions

The following is a take home exam.

- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- Each problem is worth 20pts for a total of 100 point.
- Answer question precisely and completely. Credit will be subtracted off for vague answers.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and your EE600 text book. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 743-0744 (8:00AM to 8:00PM); or office 494-0340.

Good luck.

Problem 1.(20pt)
Let X be a binary MRF such that $X_s \in \{-1, 1\}$, and

$$p(x) = \frac{1}{z} \exp\left\{ \frac{-\alpha}{2} \sum_s x_s + \frac{\beta}{2} \sum_{\{s,r\} \in C} x_s x_r \right\}$$

where C is the set of cliques for a 2-D lattice using a 4 point neighborhood.
Furthermore, we say that a discrete random object Y has an exponential distributed probability mass function if it may be written in the form

$$p(y) = c(\theta) \exp\{\theta^t t(y)\}$$

where θ is a parameter vector, and $t(y)$ is a vector valued function of y known as a sufficient statistic.
a) Assuming that $\theta = [\alpha, \beta]^t$ show that X has an exponiential distribution, and find the sufficient statistics.
b) Show that the maximum likelihood estimate of θ is a function of the sufficient statistic. Compute the ML estimate in terms of the partition function $z(\theta)$.
c) For this part, assume that $\alpha = 0$ and $\theta = \beta$. Use the results of the paper by Onsager to calculate an closed form solution for the ML estimate of θ which is valid as the size of the random field X becomes infinit.

Problem 2.(20pt)
Let $\{X_n\}_{n=0}^N$ be a 1-D discrete Markov Chain with $X_i \in \{0, 1\}$ and $X_0 = 0$ and N is odd. Also assume that

$$p(x_n | x_{n-1}) = \begin{cases} \theta & \text{if } x_n \neq x_{n-1} \\ 1 - \theta & \text{if } x_n = x_{n-1} \end{cases}$$

a) Compute the ML estimate of θ given X.

b) Compute the ML estimate of θ given X_N.

c) Compute the probability mass function $p(x_n | x_{n+1}, x_{n-1})$.

Problem 3.(20pt)

Let $\{X_n\}_{n=0}^{5}$ be a 1-D discrete Markov Chain with $X_i \in \{0,1\}$ and $X_0 = 0$. Also assume that

$$p(x_n|x_{n-1}) = \begin{cases} \frac{e^{-3}}{1+e^{-3}} & \text{if } x_n \neq x_{n-1} \\ \frac{1}{1+e^{-3}} & \text{if } x_n = x_{n-1} \end{cases}$$

Furthermore, let $\{Y_n\}_{n=1}^{5}$ be binary random variables that are conditionally independent give X with

$$p(y_n|x_n) = \begin{cases} \frac{e^{-1}}{1+e^{-1}} & \text{if } y_n \neq x_n \\ \frac{1}{1+e^{-1}} & \text{if } y_n = x_n \end{cases}$$

For the following problems, show all work.

a) Express the MAP estimate of X as the $\arg\min$ of a simple functional.

b) Use dynamic programming to compute the MAP estimate of X given $(y_1, y_2, y_3, y_4, y_5) = (0, 0, 1, 1, 1)$.

Problem 4.(20pt)

Let $\{X_n\}_{n=1}^{N}$ be a Gaussian random vector with mean 0 and covariance matrix R. Assume that R is an $N \times N$ circulant matrix, that is

$$R_{i,j} = w_{(i-j)\text{mod}N}$$

for some function w_k.

a) Show that R may be expressed in the form $R = T^H \Lambda T$ where Λ is a diagonal matrix, T is a full matrix, and T^H is the conjugate transpose of T. Specify, the entries of T and Λ. (Hint: Use the DFT transformation matrix.)

b) Find an expression for the determinant of R.

c) Assuming that $Y = X + Z$ where Z is i.i.d. Gaussian $N(0, \sigma^2)$ noise, compute the MAP estimate of X given Y.

d) Find a fast algorithm for computing the MAP estimate of X by using the FFT.

Problem 5.(20pt)

Let X be a Markov random field with $X_s \in \{0,1\}$ and lattice $S = \{(s_1, s_2) : 0 \leq s_1 \leq N-1, 0 \leq s_2 \leq N-1\}$. Let the neighbors of the point (s_1, s_2) be $\{(s_1 - 2, s_2), (s_1 - 1, s_2), (s_1 + 1, s_2), (s_1 + 2, s_2), (s_1, s_2 - 1), (s_1, s_2 + 1)\} \cap S$

a) Show that this is a legal neighborhood system.

b) Find all the cliques.

c) Give the most general form of the probability mass function for X under the assumption that the mass function is strictly positive.