EE 641 Midterm Exam

November 4, Fall 2022

Name: **Key** ______________________________

Q1: Instructions (4pt)

Rules: I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Signature: ______________________________

Q2: MAP Estimation (35pt)

Consider a zero mean N-dimensional GMRF, X, with a density function given by

$$p(x) = \frac{1}{z} \exp\left\{ -\frac{1}{2} x^t B x \right\} \ ,$$

where $B_{i,j} = a_{(i-j)\text{mod}N}$ and the row/column indices have the range $i, j \in \{0, \cdots, N-1\}$. (i.e., The rows and columns are indexed starting at 0.)

Q2.1:

What is the name given to the matrix B?

Q2.2:

Give an explicit expression for the value of z.

Q2.3:

If for $i \neq j$ we have that $B_{i,j} \neq 0$, then what do you know about the relationship between X_i and X_j?

Q2.4:

Write down an expression for the conditional expectation

$$E[X_i | X_j \, j \neq i] \ ,$$

in terms of the the function a_i.

Q2.5:

Write down an expression for the conditional distribution

$$p(x_i | x_j \, j \neq i) \ ,$$

in terms of the the function a_i.

Solution:

Q2.1:

The matrix B is called the precision matrix.

Q2.2:

$$z = (2\pi)^{N/2} |B|^{-1/2}$$

Q2.3:

If for $i \neq j$ we have that $B_{i,j} \neq 0$, then we know that X_i and X_j are neighbors. So then the conditional density, $p(x_i | x_k \text{ for } k \neq i)$, must be a function of x_j.

Q2.4:

The conditional expectation is given by

$$E[X_i | X_j \; j \neq i] = -\sum_{j \neq i} \frac{a_{(i-j) \bmod N}}{a_0} X_j \; .$$

Q2.5:

The conditional probability is given by

$$p(x_i | x_j \; j \neq i) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (x_i - \mu_i)^2 \right\}$$

where

$$\mu_i = -\sum_{j \neq i} \frac{a_{(i-j) \bmod N}}{a_0} X_j$$
$$\sigma^2 = \frac{1}{a_0} \; .$$

Q3: Augmented Lagrangian and ADMM (35pt)

Consider an inverse problem in which the map estimate is given by

$$\hat{x} = \arg\min_x \{f(x) + h(x)\}$$

where $x \in \Re^N$, and the two functions, $f : \Re^N \to \Re \cup \{\infty\}$, and $h : \Re^N \to \Re \cup \{\infty\}$, are proper closed convex functions.

Q3.1:

Write an equivalent expression for $\hat{x}$ based on the constrained optimization of two variables, (x, v).

Q3.2:

Write out an expression for the augmented Lagrangian $L(x, v; u)$ which solves the problem for the proper value of u.

Q3.3:

Let

$$(\hat{x}, \hat{v}) = \arg\min_{(x,v)} L(x, v; u) \ .$$

As u is increased, what will happen to the values of $\hat{x}$ and $\hat{v}$?

Q3.4:

Give an algorithm for solving the problem that uses the following two functions

$$F(z) = \arg\min_x \left\{ f(x) + \frac{a}{2}\|x - z\|^2 \right\} \ ,$$

$$H(z) = \arg\min_x \left\{ h(x) + \frac{a}{2}\|x - z\|^2 \right\} \ .$$

Q3.5:

If we choose,

$$h(x) = \begin{cases} \infty & \text{if } \exists i \text{ s.t. } x_i < 0 \\ 0 & \text{otherwise} \end{cases} \ ,$$

then calculate an explicit expression for $H(z)$.

Solution:

Q3.1:

An equivalent express is

$$(\hat{x}, \hat{v}) = \arg\min_{\substack{(x,v) \\ x=v}} \{f(x) + h(v)\} \ .$$

Q3.2:

$$L(x, v; u) = f(x) + h(v) + \frac{a}{2}\|x - v + u\|^2$$

Q3.3:

As u is increased, it will decrease the value of $\hat{x} - \hat{v}$. So it will tend to decrease $\hat{x}$ and increase $\hat{v}$.

Q3.4:

The algorithm is:

$u \leftarrow 0$

$v \leftarrow 0$

Repeat until converged {

$x \leftarrow F(v - u)$
$v \leftarrow H(x + u)$
$u \leftarrow u + (x - v)$

}

Return(x)

Q3.5:

In this case, we have that

$$[H(z)]_i = \begin{cases} 0 & \text{for } z_i < 0 \\ z_i & \text{otherwise} \end{cases}$$

Q4: Surrogate Functions (30pt) Consider the function

$$f(x) = \frac{1}{2}\|y - Ax\|^2 \ ,$$

with the associated quadratic surrogate function of the form

$$f(x; x') = \frac{a}{2}\|x - x'\|^2 + b^t(x - x') + c \ .$$

Q4.1:

Calculate expressions for the gradient, $\nabla f(x)$, and the Hessian, $\nabla\nabla f(x)$.

Q4.2:

Calculate expressions for c and b in the surrogate function.

Q4.3:

Calculate the matrix B so that the following expression is a Taylor series expression of $f(x)$ about the point x'.

$$f_T(x) = \frac{1}{2}(x - x')^t B(x - x') + b^t(x - x') + c$$

Q4.4:

Find an expression for a so that $f(x; x')$ is a surrogate function for $f(x)$.

Q4.5:

Give an iterative algorithm for computing the $\hat{x} = \arg\min_x f(x)$ by using the surrogate function.

Solution:

Q4.1:

The gradient is given by

$$\nabla f(x) = (Ax' - y)^t A$$

and the Hessian is given by

$$\nabla\nabla f(x) = A^t A$$

Q4.2:

Expressions are

$$c = \frac{1}{2}\|y - Ax'\|^2$$

and

$$b = A^t(Ax' - y)\ .$$

Q4.3:

$$B = A^t A$$

Q4.4:

We must choose a so that for all $x \in \Re^N$, we have that

$$x^t(aI)x \geq x^t Bx = x^t A^t Ax\ .$$

We can do this by choosing a to be the maximum eigenvalue of $A^t A$.

Q4.5:

The minimization of the surrogate function results in the following expression

$$
\begin{aligned}
F(x') &= \arg\min_x \left\{ \frac{a}{2}\|x - x'\|^2 + b^t(x - x') + c \right\} && (1)\\
&= x' - \frac{b}{a} && (2)\\
&= x' - \frac{1}{a}A^t(Ax' - y)\ . && (3)
\end{aligned}
$$

The algorithm is then:

$v \leftarrow 0$

$a \leftarrow$ `max_eigenvalue{AᵗA}`

Repeat until converged {

$\quad x \leftarrow x - \frac{1}{a}A^t(Ax - y)$

}

Return(x)

Notice that this is just gradient descent!!