EE 641 Final Exam

December 12, Fall 2022

Name: ______________________________

Q1: Instructions (4pt)

Rules: I understand that this is an open book exam that shall be done within the allotted time of 180 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam; I will not seek or accept help from any other persons other than the official proctors; and I will not use GPT-3 or any other variant of an AI response engine.

Signature: ______________________________

Q2: Generating Random Variables (10pt)

Let X be a random variable with the CDF given by

$$F(\lambda) = P\{X \leq \lambda\} \ ,$$

where F is continuous and strictly monotone increasing.

Q2.1:

Give a method for generating a new random variable, X', with the same distribution as X.

Q2.2:

Prove that if $F'(\lambda)$ is the CDF of X', then $F'(\lambda) = F(\lambda)$.

Q3: Properties of Discrete Distribution (25pt)

Consider $X = (X_0, \cdots, X_{N-1})$ where X_n are i.i.d. random variables such that

$$P\{X_n = i\} = \theta_i \ .$$

Also let $p_\theta(x)$ denote the associated family of distributions such that $\theta \in S$ where S denotes the M dimensional simplex given by

$$S = \left\{ \theta \in S : \forall i \in \{0, \cdots, M-1\} \ , \ \theta_i \geq 0 \text{ and } \sum_{i=0}^{M-1} \theta_i = 1 \right\} \ .$$

Q3.1:

Show that

$$K_i = \sum_{n=0}^{N-1} \delta(X_n - i) \ ,$$

is a sufficient statistic for the family of distributions $p_\theta(x)$.

Q3.2:

Show that $p_\theta(x)$ is an exponential distribution with natural sufficient statistics of $\{K_i\}_{i=0}^{M-1}$.

Q3.3:

Derive the maximum likelihood estimate of θ given the observations X.

Q4: ADMM Optimization (20pt)

Let $X = (X_0, \cdots, X_{N-1})$ be i.i.d. random variables with distribution

$$P\{X_n = m\} = \pi_m \ ,$$

where $\pi_i \geq 0$ and $\sum_{m=0}^{M-1} \pi_m = 1$.

Also, let $Y = (Y_0, \cdots, Y_{N-1})$ be conditionally independent discrete random variables given X with each Y_n having the conditional distribution given by

$$P\{Y_n = j | X_n = i\} = P_{i,j}$$

where $P_{i,j} \geq 0$ and $\sum_{j=0}^{M-1} P_{i,j} = 1$.

Furthermore, let $\theta = (\pi_0, P_{0,0}, \ldots, P_{0,M-1}, \cdots, \pi_{M-1}, P_{M-1,0}, \ldots, P_{M-1,M-1})$ parameterize the model.

Q4.1:

Using the statistic,

$$N_i = \sum_{n=0}^{N-1} \delta(X_n - i) \ ,$$

write out an expression for the density of X.

Q4.2:

Using the statistic,

$$K_{i,j} = \sum_{n=0}^{N-1} \delta(X_n - i)\delta(Y_n - j) \ ,$$

write out an expression for the conditional density of Y given X.

Q4.3:

Write out the negative log likelihood, $-\log p_\theta(x, y)$, in terms of the sufficient statistics N_i and $K_{i,j}$.

Q4.4:

Write out the maximum likelihood estimate of θ given the complete data, (X, Y).

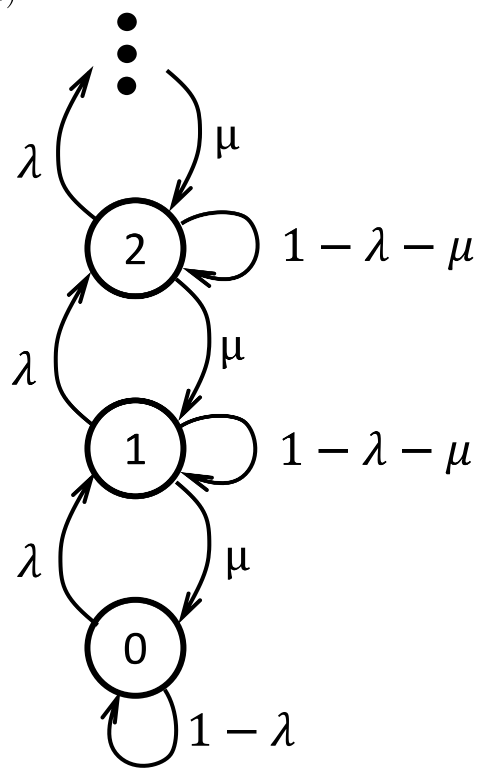
Q4.5:

Write out the explicit expression for the E-step of the EM algorithm for estimating θ given Y.

Q4.6:

Write out the explicit expression for the M-step of the EM algorithm for estimating θ given Y.

Q5: Markov Chain (25pt)



Let $\{X_n\}_{n=0}^{\infty}$ be a homogeneous Markov chain with states $\{0, 1, 2, \cdots\}$ and state-transition diagram as shown above. Furthermore, assume that $\rho = \lambda/\mu < 1$.

Q5.1:

Write out an explicit form for the state transition probabilities, $P_{i,j}$.

Q5.2:

Is there a solution to the detailed balance equations for this Markov chain? If so, give the solution.

Q5.3:

Is there a solution to the full balance equations for this Markov chain? If so, give the solution.

Q5.4:

Is the Markov chain reversible? Justify your answer.

Q5.5:

Determine the stationary distribution for the Markov chain.

Q6: Plug-and-Play Methods (25pt)

Define

$$f(x) = \frac{1}{2\sigma_y^2}\|y - Ax\|^2$$

and its associated proximal map as

$$\hat{x} = F(z) = \arg\min_x \left\{ f(x) + \frac{1}{2\sigma^2}\|x - z\|^2 \right\} \ .$$

Let $H(z)$ be a firmly non-expansive function so that

$$X \approx H(X + W) \ ,$$

where X is a typical image and $W \sim N(0, \sigma^2 I)$. Then define

$$T = (2H - I)(2F - I) \ .$$

Furthermore, assume that the fixed point problem $Tw^* = w^*$ has a unique solution denoted by w^*.

Q6.1:

Use the results of Appendix B, Properties B.5, B.3, and B.1 to prove that T is non-expansive.

Q6.2:

Give an algorithm for computing the solution to the fixed point problem $Tw^* = w^*$. Why do you know that this algorithm converges to w^*.

Q6.3:

Prove that there is a solution to the equilibrium equation

$$F(x^* - u^*) = x^*$$

$$H(x^* + u^*) = x^* \ .$$

(Hint: Reverse the argument of Section 10.3.3 page 161.)

Q6.4:

In the equilibrium equations,

$$F(x^* - u^*) = x^*$$

$$H(x^* + u^*) = x^* \ ,$$

give an interpretation for the quantities x^* and u^*.

Q6.5:

Explain how one might obtain an agent, $H(z)$?

Q6.6:

What is the advantage of this approach over more conventional MAP estimation?