EE 641 Midterm Exam

October 29, Fall 2021

Name: _____

**Q1: Instructions (4pt)**

**Rules:** I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

**Signature:** _____

**Q2: MAP Estimation** (35pt)

Consider an inverse problem in which the map estimate is given by

$$\hat{x} = \arg\min_x \left\{ \frac{1}{2\sigma^2}\|y - Ax\|^2 + \beta u(x) \right\}$$

where $y \in \Re^M$, $x \in \Re^N$, $A \in \Re^{M \times N}$, $\beta \geq 0$, and $u : \Re^N \to \Re^+$ such that $u(x)$ takes on a unique global minimum.

**Q2.1:**

Write down a specification of the forward model for this problem, i.e., a specification of the random observations $Y$ given the random unknown $X$.

**Q2.2:**

Specify the conditional probability of $Y$ given $X$.

**Q2.3:**

When $\beta = 0$, the estimate $\hat{x}$ has special characteristics.

a) What name do you use to describe this estimate?

b) What is good and bad about this estimate?

**Q2.4:**

Specify the prior probability density for $X$.

**Q2.5:**

What happens to the prior distribution as $\beta \to 0$?

**Q2.6:**

What happens to the prior distribution as $\beta \to \infty$?

**Q2.5:**

What happens to the MAP estimate, $\hat{x}$, as $\beta \to \infty$?

**Q3: Non-causal Models** (30pt)

Consider a 1D zero-mean stationary Gaussian AR process $X_n$ with prediction filter given by

$$h(n) = \rho\delta(n-1) \ ,$$

and causal prediction variance $\sigma_c^2 > 0$ where $|\rho| < 1$.

**Q3.1:**

Calculate an expression for $S_X(e^{j\omega})$, the power spectral density of $X_n$.

**Q3.2:**

Calculate an expression for the non-causal prediction filter $g(n)$.

**Q3.3:**

Calculate an expression for the non-causal prediction variance $\sigma_{nc}^2$.

**Q3.4:**

Is $X_n$ an MRF? Justify your answer.

**Q3.5:**

Define the column vector

$$Z = \begin{vmatrix} X_n \\ X_{n+1} \\ \vdots \\ X_{n+p-1} \end{vmatrix} .$$

And let $B$ be the precision matrix for $Z$ so that $Z \sim N(0, B^{-1})$.

Which entries in $B$ are zero and which are not zero?

**Q3.6:**

Specify the values of the entries in $B$.

Hint: Most of the entries are easy to specify. However, the entries in the first and last row are trickier to calculate.

**Q4: Surrogate Functions** (30pt)

Let $f(x)$ be non-negative function, and let $q(x; x')$ be a surrogate function for the minimization of $f(x)$ so that $\forall x, x' \in \Re^N$,

$$f(x') = q(x'; x')$$

$$f(x) \leq q(x; x') \ .$$

Then the majorization-minimization algorithm is given by

initialize $x^0$

initialize $k \leftarrow 0$

Repeat {

$C_k \leftarrow f(x^k)$

$x^{k+1} \leftarrow \arg\min_{x \in \Re^N} q(x; x^k)$

$k++$

}

**Q4.1:**

Sketch a figure illustrating the intuition behind the surrogate function. Make sure to label the following on your figure: i) the point $x'$; ii) the value $f(x')$; iii) the value $q(x'; x')$.

**Q4.2:**

Prove that $\forall k$, $C_k \leq C_{k-1}$.

**Q4.3:**

Prove that $C_\infty = \lim_{k \to \infty} C_k$ exists.

**Q4.4:**

For the rest of this problem, assume that

$$f(x) = |x| \ .$$

Then find a surrogate function, $q(x; x')$, with the form

$$q(x; x') = ax^2 + b \ .$$

Hint: Determine the values of $a$ and $b$ as a functions of $x'$.

**Q4.5:**

Use the result of Q4.4 above to calculate a surrogate function $g(x; x')$ for the function $g(x)$ given by

$$g(x) = \frac{1}{2}(x-2)^2 + |x| \ .$$

**Q4.6:**

Using the surrogate function of Q4.5 above, calculate the update given by

$$x^1 \leftarrow \arg\min_{x \in \Re^N} g(x; 1)$$

**Q4: Stuff** (1pt)

When interest rates go up, what happens to bond prices?