EE 641 Midterm Exam

October 29, Fall 2021

Name: **Key** ______________________________

Q1: Instructions (4pt)

Rules: I understand that this is an open book exam that shall be done within the allotted time of 120 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Signature: ______________________________

Q2: MAP Estimation (35pt)

Consider an inverse problem in which the map estimate is given by

$$\hat{x} = \arg\min_x \left\{ \frac{1}{2\sigma^2} \|y - Ax\|^2 + \beta u(x) \right\}$$

where $y \in \Re^M$, $x \in \Re^N$, $A \in \Re^{M \times N}$, $\beta \geq 0$, and $u : \Re^N \to \Re^+$ such that $u(x)$ takes on a unique global minimum.

Q2.1:

Write down a specification of the forward model for this problem, i.e., a specification of the random observations Y given the random unknown X.

Q2.2:

Specify the conditional probability of Y given X.

Q2.3:

When $\beta = 0$, the estimate $\hat{x}$ has special characteristics.

a) What name do you use to describe this estimate?

b) What is good and bad about this estimate?

Q2.4:

Specify the prior probability density for X.

Q2.5:

What happens to the prior distribution as $\beta \to 0$?

Q2.6:

What happens to the prior distribution as $\beta \to \infty$?

Q2.5:

What happens to the MAP estimate, $\hat{x}$, as $\beta \to \infty$?

Solution:

Q2.1:

$$Y = X + W \ ,$$

where $X \sim \frac{1}{z} \exp\{-u(x)\}$, $W \sim N(0, \sigma^2 I)$.

Q2.2:

$$p(y|x) = \frac{1}{(2\pi\sigma^2)^{M/2}} \exp\left\{ -\frac{1}{2\sigma^2} \|y - Ax\|^2 \right\}$$

Q2.3:

a) Maximum Likelihood Estimate

b) The ML estimate has low bias, but it typically has higher variance.

Q2.4:

$$p(x|y) = \frac{1}{z_\beta} \exp\{-\beta u(x)\} \ ,$$

where

$$z_\beta = \int_{\mathbb{R}^N} \exp\{-\beta u(x)\} \, dx \ .$$

Q2.5:

As $\beta \to 0$, the prior distribution becomes

$$p(x|y) = \frac{1}{z_\beta} \exp\{-0\} = \text{constant} \ .$$

So the prior distribution becomes uniform over the space of possible solutions.

Q2.6:

As $\beta \to \infty$, the prior distribution becomes

$$p(x|y) = \delta(x - x^*) \ ,$$

where

$$x^* = \arg\min_{x \in \mathbb{R}} u(x) \ .$$

So the prior distribution becomes concentrated about the most probable value.

Q2.5:

In this case $\hat{x} = x^*$. So the MAP estimate becomes the most probable prior value, and it is not dependent on the data.

Q3: Non-causal Models (30pt)

Consider a 1D zero-mean stationary Gaussian AR process X_n with prediction filter given by

$$h(n) = \rho\delta(n-1) \ ,$$

and causal prediction variance $\sigma_c^2 > 0$ where $|\rho| < 1$.

Q3.1:

Calculate an expression for $S_X(e^{j\omega})$, the power spectral density of X_n.

Q3.2:

Calculate an expression for the non-causal prediction filter $g(n)$.

Q3.3:

Calculate an expression for the non-causal prediction variance σ_{nc}^2.

Q3.4:

Is X_n an MRF? Justify your answer.

Q3.5:

Define the column vector

$$Z = \left| \begin{array}{c} X_n \\ X_{n+1} \\ \vdots \\ X_{n+p-1} \end{array} \right| .$$

And let B be the precision matrix for Z so that $Z \sim N(0, B^{-1})$.

Which entries in B are zero and which are not zero?

Q3.6:

Specify the values of the entries in B.

Hint: Most of the entries are easy to specify. However, the entries in the first and last row are trickier to calculate.

Solution:

Q3.1:

$$S_X(e^{j\omega}) = \frac{\sigma_c^2}{|1-\rho e^{-j\omega}|^2}$$

Q3.2:

$$g(n) = [\delta(n-1) + \delta(n+1)]\,\frac{\rho}{1+\rho^2}$$

Q3.3:

$$\sigma_{nc}^2 = \frac{\sigma_c^2}{1+\rho^2}$$

Q3.4:

Yes, any 1D AR process of order P is a 1D MRF of order P.

Q3.5:

If $|i-j| \le 1$, then $B_{i,j} \neq 0$.

If $|i-j| > 1$, then $B_{i,j} = 0$.

Q3.6:

$$\begin{aligned}
B &= \frac{1}{\sigma_c^2}\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\rho & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & -\rho & 1 & 0 \\ 0 & \dots & 0 & -\rho & 1 \end{bmatrix}\begin{bmatrix} 1 & -\rho & 0 & \dots & 0 \\ 0 & 1 & \ddots & \dots & 0 \\ 0 & 0 & \ddots & -\rho & \vdots \\ 0 & \ddots & \ddots & 1 & -\rho \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} \\
&= \frac{1}{\sigma_c^2}\begin{bmatrix} 1 & -\rho & 0 & 0 & \dots & 0 \\ -\rho & (1+\rho^2) & -\rho & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & -\rho & (1+\rho^2) & -\rho & 0 \\ 0 & \dots & 0 & -\rho & (1+\rho^2) & -\rho \\ 0 & \dots & 0 & 0 & -\rho & (1+\rho^2) \end{bmatrix}
\end{aligned}$$

Q4: Surrogate Functions (30pt)

Let $f(x)$ be non-negative function, and let $q(x; x')$ be a surrogate function for the minimization of $f(x)$ so that $\forall x, x' \in \Re^N$,

$$f(x') = q(x'; x)$$

$$f(x) \leq q(x; x') \ .$$

Then the majorization-minimization algorithm is given by

$$\begin{array}{ll} \text{initialize} & x^0 \\ \text{initialize} & k \leftarrow 0 \\ \text{Repeat } \{ & \\ & C_k \leftarrow f(x^k) \\ & x^{k+1} \leftarrow \arg\min_{x \in \Re^N} q(x; x^k) \\ & k++ \\ \} & \end{array}$$

Q4.1:

Sketch a figure illustrating the intuition behind the surrogate function. Make sure to label the following on your figure: i) the point x'; ii) the value $f(x')$; iii) the value $q(x'; x')$.

Q4.2:

Prove that $\forall k$, $C_k \leq C_{k-1}$.

Q4.3:

Prove that $C_\infty = \lim_{k\to\infty} C_k$ exists.

Q4.4:

For the rest of this problem, assume that

$$f(x) = |x| \ .$$

Then find a surrogate function, $q(x; x')$, with the form

$$q(x; x') = ax^2 + b \ .$$

Hint: Determine the values of a and b as a functions of x'.

Q4.5:

Use the result of Q4.4 above to calculate a surrogate function $g(x; x')$ for the function $g(x)$ given by

$$g(x) = \frac{1}{2}(x-2)^2 + |x| \ .$$

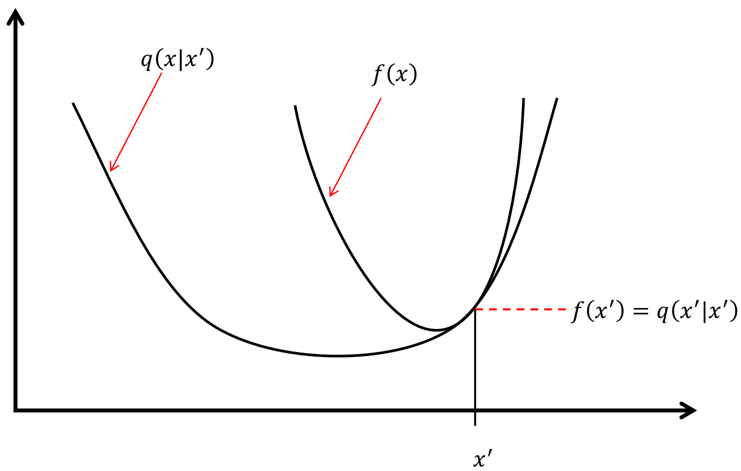
Q4.6:

Using the surrogate function of Q4.5 above, calculate the update given by

$$x^1 \leftarrow \arg\min_{x \in \Re^N} g(x; 1)$$

Solution:

Q4.1:



Q4.2:

$$C_k = f(x^k) \leq q(x^k; x^{k-1}) \leq q(x^{k-1}; x^{k-1}) = f(x^{k-1}) = C_{k-1}$$

Q4.3:

By Q4.2, we know what C_k is a monotone non-increasing function. Furthermore, since $f(x)$ is non-negative, we know what $\forall k > 0$, $C_k \geq 0$.

So since C_k is a monotone non-increasing sequence that is bounded below, we have that

$$\lim_{k \to \infty} C_k = C_\infty \in \mathbb{R} \ .$$

Q4.4:

Since we

$$\left. \frac{dq(x; x')}{dx} \right|_{x=x'} = 2ax' = \left. \frac{d|x|}{dx} \right|_{x=x'} = \text{sign}(x') \ ,$$

which implies that $a = \frac{1}{2|x'|}$, so that

$$q(x; x') = \frac{1}{2|x'|} x^2 + b \ .$$

The value of b doesn't matter, but we can calculate it anyway as

$$q(x'; x') = \frac{1}{2|x'|} (x')^2 + b = |x'| \ .$$

This implies that $b = |x'|/2$, so that we have that

$$q(x; x') = \frac{1}{2|x'|}x^2 + \frac{|x'|}{2} \ .$$

Q4.5:

$$\begin{aligned} g(x; x') &= \frac{1}{2}(x-2)^2 + q(x; x') + \text{any constant} \\ &= \frac{1}{2}(x-2)^2 + \frac{1}{2|x'|}x^2 \end{aligned}$$

Q4.6:

$$\begin{aligned} g(x; 1) &= \frac{1}{2}(x-2)^2 + \left.\frac{1}{2|x'|}x^2\right|_{x'=1} \\ &= \frac{1}{2}(x-2)^2 + \frac{1}{2}x^2 \\ &= 1 \end{aligned}$$

Q4: Stuff (1pt)

When interest rates go up, what happens to bond prices?

Solution:

They go down!

Link to article about bonds.