EE 641 Final Exam

December 14, Fall 2021

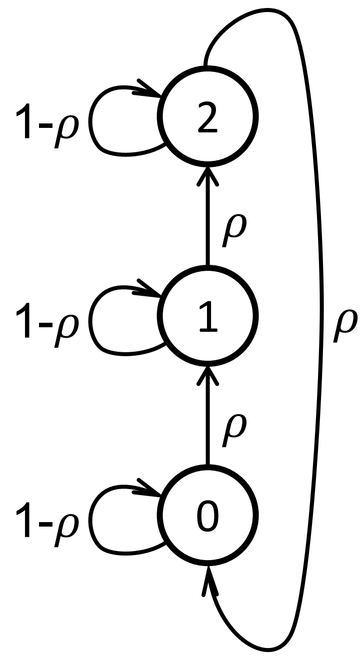
Name: ______________________________

Q1: Instructions (4pt)

Rules: I understand that this is an open book exam that shall be done within the allotted time of 180 minutes. I can use my notes, previous posted exams and exam solutions, and web resources. However, I will not communicate with any other person other than the official exam proctors during the exam, and I will not seek or accept help from any other persons other than the official proctors.

Signature: ______________________________

Q2: Markov Chain (35pt)



Let $\{X_n\}_{n=0}^{\infty}$ be a homogeneous Markov chain with states $\{0, 1, 2\}$ and state-transition diagram as shown above with $\rho \in (0, 1)$.

Q2.1:

Write out the state transition matrix, P, for X_n.

Q2.2:

Prove or disprove that X_n is irreducible.

Q2.3:

Prove or disprove that X_n is periodic.

Q2.4:

Prove or disprove that X_n is ergodic.

Q2.5:

Prove or disprove that X_n is reversible.

Q2.6:

Find the stationary distribution of X_n denoted by $P\{X_n = i\} = \pi_i$.

Q2.7:

Compute the transition matrix, Q, for the time-reversed Markov chain, X_{-n}.

Q3: EM Algorithm (25pt)

Let $\{X_n\}_{n=1}^N$ be i.i.d. random variables with distribution

$$P\{X_n = m\} = \pi_m \ ,$$

where $\sum_{m=0}^{M-1} \pi_m = 1$. Also, let Y_n be conditionally independent random variables given X_n for $n = 1, \ldots, N$ with identical exponential conditional distributions given by

$$p(y_n | x_n = m) = u(y_n) \frac{1}{\mu_m} \exp\left\{ -\frac{y_n}{\mu_m} \right\}$$

where

$$u(y) = \begin{cases} 1 \text{ if } y \geq 0 \\ 0 \text{ if } y < 0 \end{cases} .$$

Furthermore, let $\theta = (\pi_0, \mu_0, \ldots, \pi_{M-1}, \mu_{M-1})$ parameterize the model.

Q3.1:

Write out the joint density function of $(X_1, Y_1, \ldots, X_N, Y_N)$ as a function of θ.

Q3.2:

Write out the joint density function for $(Y_1, \ldots, Y_N)$ as a function of θ.

Q3.3:

Write out an expression for the maximum likelihood estimate of θ given $(X_1, Y_1, \ldots, X_N, Y_N)$.

Q3.4:

Write out an expression for the E-step of the EM update of θ given $(Y_1, \ldots, Y_N)$. (Hint: Compute the posterior conditional expectation of the natural sufficient statistics of the distribution.)

Q3.5:

Write out an expression for the M-step of the EM update of θ given $(Y_1, \ldots, Y_N)$.

Q4: ADMM Optimization (20pt)

Consider the following MAP optimization problem

$$\hat{x} = \arg\min_{x} \{f(x) + h(x)\} \ ,$$

where $x \in \Re^N$.

Q4.1:

Write out the expression for the associated constrained optimization problem produced by splitting.

Q4.2:

Write out the associated augmented Lagrangian cost function $L(x, v; a, u)$.

Q4.3:

Write out pseudo-code for the augmented Lagrangian algorithm used for solving the MAP optimization problem.

Q4.4:

Write out pseudo-code for the ADMM algorithm for solving the MAP optimization problem.

Q5: Proximal Maps (25pt)

Consider the proximal maps given by

$$\hat{x} = F(z) = \arg\min_x \left\{ f(x) + \frac{1}{2\sigma^2}\|x - z\|^2 \right\}$$

$$\hat{x} = H(z) = \arg\min_x \left\{ \frac{1}{2\sigma^2}\|z - x\|^2 + h(x) \right\} ,$$

where f and h are continuously differentiable and convex.

Then our goal is to solve for the equilibrium conditions given by

$$F(x^* + u^*) = x^*$$

$$H(x^* - u^*) = x^* .$$

Q5.1:

Write out the basic plug-and-play ADMM algorithm for solving the equilibrium conditions.

Q5.2:

Prove that the values (x^*, u^*) that solve the equilibrium conditions also minimize the function $f(x) + h(x)$.

Q5.3:

Give an interpretation for the form of $H(z)$ as a MAP estimate. Provide an interpretation for each of the two terms in the cost function being minimized.

Q5.4:

Give an interpretation for the form of $F(z)$ as a MAP estimate. Provide an interpretation for each of the two terms in the cost function being minimized.

Q5.5:

If we were to replace $H(z)$ by a function that is trained using machine learning methods, then how should it be trained?