EE 641 Midterm Exam

October 18, Fall 2019

Name: ______________________________

Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 100 points.
- You may not use any notes, textbooks, or calculators.
- You are allowed up to 55 minutes to complete the exam.

Good luck.

Problem 1. (32pt) Let $X \sim N(0, R)$ where R is a $p \times p$ symmetric positive-definite matrix. Further define the precision matrix, $B = R^{-1}$, and use the notation

$$B = \left[\begin{array}{cc} 1/\sigma^2 & A \\ A^t & C \end{array} \right] ,$$

where $A \in \mathbb{R}^{1\times(p-1)}$ and $C \in \mathbb{R}^{(p-1)\times(p-1)}$.

a) Calculate the marginal density of X_1, the first component of X, given the components of the matrix R.

b) Calculate the conditional density of X_1 given all the remaining components, $Y = [X_2, \cdots, X_p]^t$.

c) What is the conditional mean and covariance of X_1 given Y?

Problem 2. (32pt) Consider the optimization problem

$$\hat{x} = \arg \min_{x \in \mathbb{R}^N} \left\{ ||y - Ax||_\Lambda^2 + x^t B x \right\}$$

where A is a nonsingular $N \times N$ matrix, B is a positive-definite $N \times N$ matrix, and Λ is a diagonal and positive-definite matrix.

a) Derive a closed form expression for the solution.

b) Calculate an expression for the gradient descent update using step size $\mu \geq 0$.

c) Calculate an expression for the coordinate descent update.

Problem 3. (36pt) For the following problem, consider the MAP cost function given by

$$f(x) = \frac{1}{2}||y - Ax||_\Lambda^2 + \sum_{\{s,r\}\in\mathcal{P}} b_{s,r}\rho(x_s - x_r) \ . \tag{1}$$

where $y \in \Re^N$, $x \in \Re^N$, $A \in \Re^{N\times N}$ has rank N, Λ is positive-definite, and $\rho(\Delta)$ is a positive convex function of Δ. Also, define the subevel set $\mathcal{A}_\alpha$ to be

$$\mathcal{A}_\alpha = \left\{ x \in \mathbb{R}^N : f(x) \leq \alpha \right\} \ ,$$

and define the inverse image of a set $S \subset \Re$ to be

$$f^{-1}(S) = \{x \in \Re^N : f(x) \in S\} \ .$$

For this problem you can use the following theorems:

T1: A set in $\Re^N$ is compact if and only if it is closed and bounded.

T2: If f is a continuous function and S is closed, then the inverse images $f^{-1}(S)$ is closed.

a) Prove that for all $\alpha \in \mathbb{R}$ the subevel set $\mathcal{A}_\alpha$ is closed.
b) Prove that there exists an $\alpha \in \mathbb{R}$ such that the subevel set $\mathcal{A}_\alpha$ is non-empty and compact.
c) Prove there exists a MAP estimate, $\hat{x}$, so that $\forall x \in \Re^N$, $f(\hat{x}) \leq f(x)$.
d) Prove that the MAP estimate is unique.