EE 641 Final Exam

December 9, Fall 2019

Name: ______________________________

Instructions

- This exam contains 4 problems worth a total of 100 points.
- You may have up to 120 minutes to take the exam.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (25pt)

Let $f(x)$ and $q(x; x')$ both be continuously differentiable and convex function of x, such that $\forall x', x \in \mathbb{R}^N$,

$$\begin{aligned} f(x') &= q(x'; x') && (1) \\ f(x) &\leq q(x; x') \; . && (2) \end{aligned}$$

Then using the initial state $x^{(k)}$, we can compute an updated state, $x^{(k+1)}$, using the following iteration.

$$x^{(k+1)} = \arg \min_{x \in \mathbb{R}^N} \left\{ q(x; x^{(k)}) \right\} \quad (3)$$

where $\forall x$,

$$q(x^{(k+1)}; x^{(k)}) \leq q(x; x^{(k)}) \; .$$

a) Prove that $f(x^{(k+1)}) \leq f(x^{(k)})$.

b) Draw a figure illustrating why a) is true.

c) Prove that if $x^{(k+1)} = x^{(k)}$, then $\forall x$, $f(x^{(k+1)}) \leq f(x)$.

Problem 2. (25pt)

Consider the general problem of convex optimization with a positivity constraint given by

$$\hat{x} = \arg\min_{x} \{f(x) + h(x)\} \ ,$$

where $f : \mathbb{R}^N \rightarrow \mathbb{R}$ and $h : \mathbb{R}^N \rightarrow \mathbb{R}$ are both convex functions on $\mathbb{R}^N$.

a) Use variable splitting to derive a constrained optimization problem that is equivalent to this problem.

b) Formulate the augmented Lagrangian for the constrained optimization problem of a) and give the iterative algorithm for solving the augmented Lagrangian problem.

c) Write out an expression for the two proximal map functions $F(x)$ and $H(x)$ corresponding the the function $f(x)$ and $h(x)$.

d) Write out the ADMM algorithm for solving this problem in terms of the proximal maps $F(x)$ and $H(x)$.

Problem 3. (25pt)

Let $\{X_n\}_{n=1}^N$ be i.i.d. random variables with distribution

$$P\{X_n = m\} = \pi_m \ ,$$

where $\sum_{m=0}^{M-1} \pi_m = 1$. Also, let Y_n be conditionally independent random variables given X_n, with Poisson conditional distribution

$$p(y_n | x_n = m) = \frac{\lambda_m^{y_n} e^{-\lambda_m}}{y_n!} \ .$$

a) Write out the density function for the vector Y.

b) What are the natural sufficient statistics for the complete data (X, Y)?

c) Give an expression for the ML estimate of the parameter

$$\theta = (\pi_0, \lambda_0, \cdots, \pi_{M-1}, \lambda_{M-1}) \ ,$$

given the complete data (X, Y).

d) Give the EM update equations for computing the ML estimate of the parameter $\theta = (\pi_0, \lambda_0, \cdots, \pi_{M-1}, \lambda_{M-1})$ given the incomplete data Y.

Problem 4. (25pt)

Let X_n be a sequence of multivariate random vectors that form a homogeneous Markov Chain. More specifically, for each n, let $X_n = (X_{n,0}, X_{n,1}, \cdots, X_{n,M-1})$ where $X_{n,m} \in \{0, \cdots, K-1\}$.

Furthermore, let $p(x) > 0$ be any probability density function defined over the set $x \in \{0, \cdots, K-1\}^M$, and let $p_m(x_m | x_i \text{ for } i \neq m) > 0$ be its associated conditional density functions.

Given these definitions, the rule for generating X_n given X_{n-1} is given by:

Step 1: Generate a uniformly distributed random variable J on the set $\{0, \cdots, M-1\}$.

Step 2: Generate an independent random variable $W \sim p_J(x_J | X_i \text{ for } i \neq J)$.

Step 3: For $i \neq J$, set $X_{n,i} \leftarrow X_{n-1,i}$; and for $i = J$, set $X_{n,i} \leftarrow W$.

a) Show that the Markov chain has a finite number of states.

b) Show that the Markov chain is irreducible.

c) Show that the Markov chain is aperiodic.

d) Prove that Markov chain is ergodic with asymptotic distribution $p(x)$.

e) Intuitively, why does this make sense?