EE 641 Midterm Exam

October 12, Fall 2018

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Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 100 points.
- You may not use any notes, textbooks, or calculators.
- You are allowed up to 55 minutes to complete the exam.

Good luck.

Problem 1. (35pt)

Let X and W be independent Gaussian random vectors of dimension p such that $X \sim N(0, R_x)$ and $W \sim N(0, R_w)$, and let θ be a deterministic vector of dimension p.

a) First assume that $Y = \theta + W$, and calculate the ML estimate of θ given Y .

For the next parts, assume that $Y = X + W$.

b) Calculate an expression for $p_{x|y}(x|y)$, the conditional density of X given Y.

c) Calculate the MMSE estimate of X when $Y = X + W$.

d) Calculate an expression for the conditional variance of X given Y.

a) $\hat{\theta} = \arg\max_{\theta} -\frac{1}{2}\|Y-\theta\|^2 = \arg\min_{\theta} \|Y-\theta\|^2$

$\hat{\theta} = Y$

b) $\log p(x|y) = \log p(x,y) + c(y)$

$= \log p(y|x) + \log p(x) + c'(y)$

So if $E[X|Y] = AY$

$Var[X|Y] = R_{X|Y}$

then

$-\log p(x|y) = \frac{1}{2}(X-AY)^t R_{X|Y}^{-1} (X-AY)$

$= \frac{1}{2}(X-Y)^t R_w^{-1}(X-Y) + \frac{1}{2} X^t R_x^{-1} X + c(y)$

$\Rightarrow \quad R_{X|y}^{-1} = R_w^{-1} + R_x^{-1}$

$R_{X|y} = (R_w^{-1} + R_X^{-1})^{-1}$

also $\Rightarrow \quad R_{X|y}^{-1}(X-Ay) = R_w^{-1}(X-Y) + R_X^{-1}X$

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$$R_{x|y}^{-1} x - R_{x|y}^{-1} A y = (R_w^{-1} + R_x^{-1}) x - R_w^{-1} y$$

$$\forall y \quad R_{x|y}^{-1} A y = R_w^{-1} y$$

$$R_{x|y}^{-1} A = R_w^{-1}$$

$$A = (R_w^{-1} + R_x^{-1})^{-1} R_w^{-1}$$

$$= \left[R_w (R_w^{-1} + R_x^{-1}) \right]^{-1}$$

$$= \left[(R_w + R_x) R_x^{-1} \right]^{-1}$$

$$= R_x (R_w + R_x)^{-1}$$

$$p(x|y) = \frac{1}{(2\pi)^{p/2}} |R_{x|y}|^{-1/2} \exp\left\{ -\frac{1}{2} (x - Ay)^t R_{x|y}^{-1} (x - y) \right\}$$

when $R_{x|y} = (R_w^{-1} + R_x^{-1})^{-1}$

$$A = R_x (R_w + R_w)^{-1}$$

c) MMSE is $\hat{X} = AY$

d) $R_{x|y} = (R_w^{-1} + R_x^{-1})^{-1}$

Problem 2. (35pt)
Let $X \in \Re^N$ be a Gaussian random vector with $X \sim N(0, B^{-1})$, and let

$$Y = AX + W$$

where $W \sim N(0, \sigma^2 I)$. Find expressions for the following.

a) $\mathbb{E}[XX^t]$

b) $\mathbb{E}[X_i | X_j \text{ for } j \neq i]$

c) The MAP of X given Y.

d) The MMSE of X given Y.

a) $E[XX^t] = B^{-1}$

b) $E[x_i | x_j \, j \neq i] = \sum_j \frac{-B_{ij}}{B_{ii}} x_j$

c) $\hat{X} = \arg\max_x \log p(x, y)$

$= \arg\min_x \Big\{ \underbrace{\frac{1}{2\sigma^2} \|Y - Ax\|^2 + \frac{1}{2} x^t B x}_{f(x)} \Big\}$

$\nabla_x f(x) = 0$

$\Rightarrow -\frac{1}{\sigma^2} A^t (Y - Ax) + Bx = 0$

$A^t A x - A^t Y + \sigma^2 B x = 0$

$(A^t A + \sigma^2 B) x = A^t Y$

$\hat{x} = (A^t A + \sigma^2 B)^{-1} A^t Y$

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d) MMSE = MAP

$$\hat{X} = (A^\dagger A + \sigma^2 B)^{-1} A^\dagger y$$

Problem 3. (30pt)

Let $B \in \Re^{n \times n}$ be the precision matrix (i.e., the inverse covariance) of a zero-mean GMRF, and let $\mathcal{P}$ be the set of all unique unordered pairs, i.e., $\mathcal{P} = \{\{i, j\} : 1 \le i, j \le n\}$.
Then consider the following equality

$$LHS \quad = \quad x^t B x = \sum_{i=1}^{n} a_i x_i^2 + \sum_{\{i,j\} \in \mathcal{P}} b_{i,j} |x_i - x_j|^2 \quad = RHS$$

Find expressions for the values of the coefficients a_i and $b_{i,j}$ and prove that the equality holds for all $x \in \Re^n$.

$$\nabla LHS = 2BX$$

$$\frac{\partial LHS}{\partial X_i} = 2 \sum_j B_{ij} X_j$$

$$\frac{\partial RHS}{\partial X_i} = 2 a_i X_i + 2 \sum_j b_{ij} (X_i - X_j)$$

$$= 2\left(a_i + \sum_j b_{ij}\right) X_i - 2 \sum_j b_{ij} X_j$$

$\Rightarrow$

$$2 \sum_j B_{ij} X_j = 2\left(a_i + \sum_j b_{ij}\right) X_i - 2 \sum_j b_{ij} X_j$$

$$= \quad a_i + \sum_j b_{ij} = 0$$

$$B_{ij} = -b_{ij}$$

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$$\boxed{\begin{aligned} b_{ij} &= -B_{ij} \\ a_i &= \sum_j B_{ij} \end{aligned}}$$

since

$$LHS(x=0) = 0$$

$$= RHS(x=0) = 0$$

and $\nabla LHS = \nabla RHS$

$$\Rightarrow LHS = RHS \ \forall x$$