EE 641 Final Exam

Fall 2017

Name: ________________________________

Instructions

- This exam contains 4 problems worth a total of 100 points.
- You may have up to 120 minutes to take the exam.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (25pt) Consider the homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with parameters

$$\begin{aligned} \tau_j &= P\{X_0 = j\} \\ P_{i,j} &= P\{X_n = j | X_{n-1} = i\} \end{aligned}$$

where $i, j \in \{0, \cdots, M-1\}$. Furthermore, assume that the transition parameters are given by

$$P_{i,j} = \begin{cases} 1/2 & \text{if } j = (i+1)\text{mod}M \\ 1/2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

a) Write out the transition matrix P for the special case of $M = 4$. (But solve the remaining problems for any M.)

b) Is the Markov chain irreducible? Prove or give a counter example.

c) Is the Markov chain periodic? Prove or give a counter example.

d) Is the Markov chain ergodic? Prove or give a counter example.

e) Determine the value of the following matrix

$$\lim_{n \to \infty} P^n$$

f) Is the Markov chain reversible? Prove or give a counter example.

Problem 2. (25pt) Let $\rho(x)$ for $x \in \mathbb{R}$ be the Huber function for $T = 1$ given by

$$\rho(x) = \begin{cases} \frac{x^2}{2} & \text{for } |x| < 1 \\ |x| - \frac{1}{2} & \text{for } |x| \geq 1 \end{cases} .$$

The objective of this problem is to determine a surrogate function, $Q(x; x')$, for $\rho(x)$.

a) Use the symmetric bound method to determine $Q(x; x')$.

b) From $Q(x; x')$, determine the function $q(x; x')$, so that $q(x'; x') = \rho(x')$.

c) Plot the functions $\rho(x)$ and $q(x; x')$ for $x' = 4$ and indicate the location of the minimum

$$x'' = \arg\min_{x \in \mathbb{R}} q(x; 4)$$

d) What is the advantage of the symmetric bound methods?

Problem 3. (25pt) Let $\{X_n\}_{n=1}^N$ be a i.i.d. random varibles with distribution

$$P\{X_n = m\} = \pi_m \ ,$$

where $\sum_{m=0}^{M-1} \pi_m = 1$. Also, let Y_n be conditionally independent random variables given X_n, with Poisson conditional distribution

$$p(y_n|x_n = m) = \frac{\lambda_m^{y_n} e^{-\lambda_m}}{y_n!} \ .$$

a) Write out the density function for the vector Y.

b) What are the natural sufficient statistics for the complete data (X, Y)?

c) Give an expression for the ML estimate of the parameter

$$\theta = (\pi_0, \lambda_0, \cdots, \pi_{M-1}, \lambda_{M-1}) \ ,$$

given the complete data (X, Y).

d) Give the EM update equations for computing the ML estimate of the parameter $\theta = (\pi_0, \lambda_0, \cdots, \pi_{M-1}, \lambda_{M-1})$ given the incomplete data Y.

Problem 4. (25pt) Let X_s be a 2D random field with an Ising distribution given by

$$p(x) = \frac{1}{z(\beta)} \exp \left\{ -\beta \sum_{\{r,s\} \in \mathcal{C}} \delta(x_r \neq x_s) \right\} ,$$

and let the function $v(x_s, x_{\partial s})$ be defined by

$$v(x_s, x_{\partial s}) \triangleq \sum_{r \in \partial s} \delta(x_s \neq x_r) \ .$$

Then do the following:

a) Derive an expression for the conditional distribution of a pixel given its neighbors.

b) Sketch the conditional probability $p(x_s = 1 | x_{i \neq s})$ as a function of $v(1, x_{\partial s})$ with $\beta = 1$.

c) What would a "typical" sample of the X look like when $\beta = -2$?