EE 641 Final Exam

Fall 2015

Name: Key

Instructions

- This exam contains 4 problems worth a total of 100 points.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (25pt)

Consider the function

$$f(x) = |x - x_r|^{1.1} \ ,$$

for $x \in \mathbb{R}$.

a) Sketch a plot of $f(x)$ when $x_r = 1$.

b) Sketch a good surrogate function, $f(x; x')$, for $x_r = 1$ and $x' = 2$.

c) Determine a general expression for the surrogate function $f(x; x')$ that works for any value of x_r and x' such that $x_r \neq x'$.

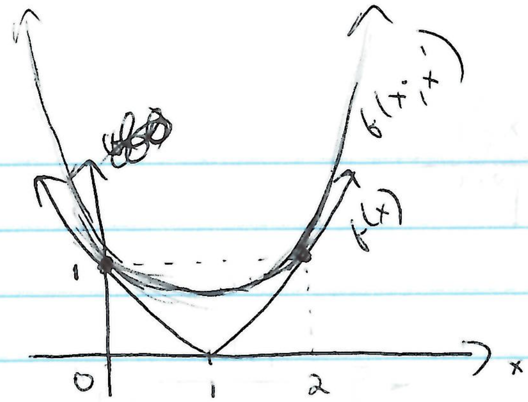
d) Assuming the objective is to minimize the expression

$$f(x) = \sum_{r \in \partial s} |x - x_r|^{1.1} \ ,$$

for $x \in \mathbb{R}$, specify an iterative algorithm in terms of the surrogate function $f(x; x')$ that will converge to the global minimum of the function.

①

a) + b)



c) $f(x;x') = \frac{a}{2}(x-x_r)^2$

$$\frac{df(x)}{dx}\Big|_{x=x'} = 1.1\,|x'-x_r|^{0.1} = a(x'-x_r)$$

then

$$1.1\,|x'-x_r|^{0.1} = a(x'-x_r)$$

then

$$a = \frac{1.1}{|x'-x_r|^{0.9}}$$

then

$$f(x;x') = \frac{1.1\,(x-x_r)^2}{2|x'-x_r|^{0.9}}$$

d) $f(x) = \sum_{rets} |x-x_r|^{1.1}$

we may substitute our surrogate formulation:

$$f(x;x') = \sum_{rets} \frac{1.1\,(x-x_r)^2}{2|x'-x_r|^{0.9}}$$

We can find a closed-form sol'n: (on back)

$$\frac{d f(x; x')}{dx} = \sum_{r \neq s} \frac{1.1(x - x_r)}{|x' - x_r|^{0.9}}$$

set $= 0$

$$\Rightarrow \quad 0 = \sum_{r \neq s} \frac{1.1 \hat{x}}{|x' - x_r|^{0.9}} - \sum_{r \neq s} \frac{1.1 x_r}{|x' - x_r|^{0.9}}$$

$$\boxed{\hat{x} = \frac{\sum_{r \neq s} |x' - x_r|^{0.9} \cdot x_r}{\sum_{r \neq s} |x' - x_r|^{0.9}}}$$

Then an algorithm meeting the given specs is:

for $k = 1 : \text{Iters}$ {

  for $s \in S$ {

  - $x' \leftarrow x_s$
  - find $f(x; x')$
  - $\hat{x} \leftarrow \arg\min_x f(x; x')$

  (in our case,

$$\hat{x} = \frac{\sum_{r \neq s} |x' - x_r|^{0.9} \cdot x_r}{\sum_{r \neq s} |x' - x_r|^{0.9}}$$

  }

  if (converged) {
    break
  }

}

Problem 2. (25pt)
Consider the problem

$$\hat{x} = \arg\min_{x \geq 0} f(x) \ ,$$

where $f : \mathbb{R}^N \to \mathbb{R}$ is a convex function and $x \geq 0$ denotes a positivity constraint on x. In order to remove the constraint, we may define the proper, closed, convex function

$$g(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ \infty & \text{if } x < 0 \end{cases} \ .$$

Then the minimum is given by the solution to the unconstrained optimization problem

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \{ f(x) + g(x) \} \ . \tag{1}$$

Using this formulation, do the following.
a) Use variable splitting to derive a constrained optimization problem that is equivalent to equation (1).
b) Formulate the augmented Lagrangian for this constrained optimization problem, and give the iterative algorithm for solving the augmented Lagrangian problem.
c) Use the ADMM approach to formulate an iterative algorithm for solving the augmented Lagrangian.
d) Simplify the expressions for the ADMM updates and give the general simplified ADMM algorithm for implementing positivity constraints in convex optimization problems.

②

a) $\hat{x} = \underset{\substack{x, z \in \mathbb{R}^N \\ x = z}}{\arg\min} \{ f(x) + g(z) \}$

b) $L(x, z; a, u) = f(x) + g(z) + \frac{a}{2} \|x - z + u\|^2$

Repeat {

$\hat{x}, \hat{z} \leftarrow \underset{\substack{x \geq 0 \\ z \in \mathbb{R}^K}}{\arg\min} \{ L(x, z; a, u) \}$

$u \leftarrow u + \hat{x} - \hat{z}$

}

c) Repeat {

$\hat{x} \leftarrow \underset{x \in \Omega}{\arg\min} \{ f(x) + \frac{a}{2} \| x - (\hat{z} - u) \|^2 \}$

$\hat{z} \leftarrow \underset{z \in \mathbb{R}^K}{\arg\min} \{ \frac{1}{2} \| z - (\hat{x} + u) \|^2 + \frac{1}{a} g(z) \}$

$u \leftarrow u + \hat{x} - \hat{z}$

}

d) Repeat {

$\hat{x} \leftarrow \underset{x \in \mathbb{R}^N}{\arg\min} \{ f(x) + \frac{a}{2} \| x - (\hat{z} - u) \|^2 \}$

$\hat{z} \leftarrow \max \{ \hat{x} + u, 0 \}$

$u \leftarrow u + \hat{x} - \hat{z}$

}

Problem 3. (25pt)

Let X_n be N i.i.d. random variables with $P\{X_n = i\} = \pi_i$ for $i = 0, \cdots, M-1$. Also, assume that Y_n are conditionally independent Gaussian random variables given X_n and that the conditional distribution of Y_n given X_n is distributed as $N(\mu_{x_n}, \gamma_{x_n})$. Derive an EM algorithm for estimating the parameters $\{\pi_i, \mu_i, \gamma_i\}_{i=0}^{M-1}$ from the observations $\{Y_n\}_{n=1}^{N}$.

(3)

Compute Statistics

MLE

$$N_i = \sum_{n=1}^{N} \delta(x_n - i)$$

$$b_i = \sum_{n=1}^{N} \delta(x_n - i) y_n \quad \Rightarrow$$

$$S_i = \sum_{n=1}^{N} y_n y_n^t \delta(x_n - i)$$

$$\hat{\pi}_i = \frac{N_i}{N}$$

$$\hat{\mu}_i = \frac{b_i}{N_i}$$

$$\hat{\gamma}_i = \frac{S_i}{N_i} - \mu_i^2$$

E-STEP

$$\bar{N}_i^{(k+1)} = E[N_i \mid Y_n, \theta^k]$$

$$= E[\sum_n \delta(x_n - i) \mid Y_n, \theta^k]$$

$$= \sum_n E[\delta(X_n - i) \mid Y_n, \theta^k]$$

$$= \sum_n P\{X_n = i \mid Y_n, \theta^k\}$$

$$\triangleq P_{n,i}^k$$

$$\bar{b}_i^{(k+1)} = \sum_{n=1}^{N} y_n P_{n,i}^{(k)}$$

$$\bar{S}_i^{(k+1)} = \sum_n y_n y_n^t P_{n,i}^{(k)}$$

M-STEP

$$\hat{\pi}_i^{(k+1)} = \frac{\bar{N}_i^{(k+1)}}{N}$$

$$\hat{\mu}_i^{(k+1)} = \frac{\bar{b}_i^{(k+1)}}{\bar{N}_i^{(k+1)}}$$

$$\hat{\gamma}_i^{(k+1)} = \frac{\bar{S}_i^{(k+1)}}{\bar{N}_i^{(k+1)}} - \left(\hat{\mu}_i^{(k+1)}\right)^2$$

Problem 4. (25pt)

Consider the homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with parameters

$$\begin{aligned} \tau_j &= P\{X_0 = j\} \\ P_{i,j} &= P\{X_n = j | X_{n-1} = i\} \end{aligned}$$

where $i, j \in \{0, \cdots, M-1\}$. Furthermore, assume that the transition parameters are given by

$$P_{i,j} = \begin{cases} 1/2 & \text{if } j = (i+1) \bmod M \\ 1/2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

a) Write out the transition matrix P for the special case of $M = 4$. (But solve the remaining problems for any M.)

b) Is the Markov chain irreducible? Prove or give a counter example.

c) Is the Markov chain periodic? Prove or give a counter example.

d) Is the Markov chain ergodic? Prove or give a counter example.

e) Determine the value of the following matrix

$$\lim_{n \to \infty} P^n$$

f) Is the Markov chain reversible? Prove or give a counter example.

④

a) $P_{ij} = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{bmatrix}$

b) i, j states communicate $\forall i, j \Rightarrow$ Irreducible Markov Chain

state 1	transitions	to	state 2	w/	prob	1/2
" 2	"	"	" 3	"	"	"
" 3	"	"	" 4	"	"	"
" 4	"	"	" 1	"	"	"

Thus it is possible for any node to change from any state to any other state in finite time, thus our Markov chain is irreducible //

c) Not periodic. $\forall$ states, $P\left(\begin{smallmatrix}\text{state stays}\\ \text{the same}\end{smallmatrix}\right) \neq 0$.

d) Yes it is ergodic

finite # of states ✓

irreducible ✓

aperiodic ✓

e) Full-balance equations give

$$\pi_j = \lim_{n\to\infty} [P^n]_{ij} > 0$$

$$\pi^\infty P = \pi^\infty, \quad \sum_{i=0}^{M-1} (\pi^\infty)_i = 1$$

then $(\pi^\infty)_i = \frac{1}{M}$

then $P^\infty_{ij} = \frac{1}{M}$ //

f) $\pi_i P_{i,i+1} = \frac{1}{M}\frac{1}{2} \neq \pi_{i+1} P_{i+1,i} = \frac{1}{M} \cdot 0$

Not Reversible