EE 641 Midterm Exam

October 16, Fall 2015

Name: ______________________________

Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 100 points.
- You may not use any notes, textbooks, or calculators.
- You are allowed up to 55 minutes to complete the exam.

Good luck.

Problem 1. (30pt)

Let X be an MRF with the following distribution

$$p(x) = \frac{1}{z} \exp \left\{ - \sum_{\{s,r\} \in \mathcal{P}} b_{s,r} \rho \left(\frac{x_s - x_r}{\sigma} \right) \right\} ,$$

where $\mathcal{P}$ is the set of neighboring pixel pairs. Derive a simplified expression for the conditional distribution of X_s given X_r for $r \neq s$.

Problem 2. (35pt)

Let X and W be independent Gaussian random vectors of dimension p such that $X \sim N(0, R_x)$ and $W \sim N(0, R_w)$, and let θ be a deterministic vector of dimension p.

a) First assume that $Y = \theta + W$, and calculate the ML estimate of θ given Y .

b) For the next parts, assume that $Y = X + W$, and calculate an expression for $p_{x|y}(x|y)$, the conditional density of X given Y.

c) Calculate the MMSE estimate of X when $Y = X + W$.

d) Calculate an expression for the conditional variance of X given Y.

Problem 3. (35pt)

Let A be an $N \times N$ circulant matrix, so that $A_{i,j} = h_{(i-j)\,mod\,N}$, for some real-valued function h_n. In order to simplify notation, we will assume all matrices in this problem are indexed from 0 to $N-1$, rather than from 1 to N. Using this convention, define the following matrix for $0 \leq m, n < N$,

$$T_{m,n} = \frac{1}{\sqrt{N}} e^{-j\frac{2\pi mn}{N}} \; .$$

Then T is known as the N dimensional orthonormal discrete Fourier transform (DFT).

a) Show that the DFT is an orthonormal transform by showing that the columns of the matrix are orthogonal and normal. So formally this means for $0 \leq m, k < N$

$$\sum_{n=0}^{N-1} T_{m,n} T^*_{k,n} = \delta_{m-k} \; .$$

b) Show that inverse transformation is given by

$$\left[T^{-1}\right]_{m,n} = T^*_{n,m}$$

where T^{-1} is the inverse DFT.

c) Show that $\Lambda = TAT^{-1}$ is a diagonal matrix with entries given by the DFT of the function h_n. That is $\Lambda = \text{diag}\left\{\lambda_0, \cdots, \lambda_{N-1}\right\}$ where $\lambda_m = \sqrt{N} \sum_{n=0}^{N-1} T_{m,n} h_n$.

d) Show that the eigenvalues of A are the diagonal entries of Λ and that the eigenvectors are the corresponding columns of T^{-1}.