EE 641 Final Exam

Fall 2014

Name: ______________________________

Instructions

- This exam contains 4 problems worth a total of 105 points.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (25pt)

Let x and y be vectors in R^N.

a) Calculate a closed form expression for the ICD update when the MAP cost function has the form.

$$f(x; y) = ||y - x||^2 + ||x||_1$$

where $||x||_1$ is the L_1 norm of x.

b) Prove that $f(x; y)$ is a strictly convex function of x.

c) Prove that this function takes on a unique global minimum.

Name: ______________________________

Problem 2. (30pt) Consider the homogeneous Markov chain $\{X_n\}_{n=0}^{\infty}$ with parameters

$$\begin{aligned} \tau_j &= P\{X_0 = j\} \\ P_{i,j} &= P\{X_n = j, X_{n-1} = i\} \end{aligned}$$

where $i, j \in \{0, \cdots, M-1\}$. Furthermore, assume that the transition parameters are given by

$$P_{i,j} = \begin{cases} 1/2 & \text{if } j = (i+1) \bmod M \\ 1/2 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}$$

a) Write out the transition matrix P for the special case of $M = 4$. (But solve the remaining problems for any M.)

b) Is the Markov chain irreducible? Prove or give a counter example.

c) Is the Markov chain periodic? Prove or give a counter example.

d) Is the Markov chain ergodic? Prove or give a counter example.

e) Determine the value of the following matrix

$$\lim_{n \to \infty} P^n$$

f) Is the Markov chain reversible? Prove or give a counter example.

Name: ________________________________

Problem 3. (25pt) Let X_n be a sequence of i.i.d. random variables for $n = 1, \cdots, N$, and let Y_n be a sequence of random variables that are conditionally independent and exponentially distributed with mean μ_i for $i = 0, \cdots, M-1$ given the corresponding values of X_n. More specifically, the distributions of X and Y are given by

$$\begin{aligned} P\{X_n = i\} &= \pi_i \\ p(y_n|x_n) &= \frac{1}{\mu_{x_n}} e^{-y_n/\mu_{x_n}} u(y_n) \end{aligned}$$

where $i \in \{0, \cdots, M-1\}$ and $u(\cdot)$ is the unit step function. Furthermore, let $\theta = [\pi_0, \mu_0, \cdots, \pi_{M-1}, \mu_{M-1}]$ be the parameter vector of the distribution.

a) Derive a closed form expression for the ML estimate of θ given both X and Y (i.e., the complete data).

b) Find the natural sufficient statistics for the exponential distribution of (X, Y).

c) Derive the EM algorithm for computing the ML estimate of θ from Y. (Feel free to use the shortcuts we learned in class.)

Name: ______________________________

Problem 4. (25pt) Consider the positive random variable X with density

$$p(x) = \frac{1}{\sigma z(p)} \exp\left\{ -\frac{1}{p\sigma^p} |x|^p \right\} ,$$

with $p = 1.2$ and $\sigma = 1$. Consider the case of a Hastings-Metropolis simulation algorithm for sampling from the distribution of $p(x)$ with the proposals generated as $W \sim q(w)$ where

$$q(w) = \frac{1}{2\alpha} \exp\left\{ -\frac{1}{\alpha} |w| \right\} ,$$

where $\alpha = 1$.

a) Sketch the density function for $p(x)$ and the proposal density $q(w)$. How do they compare?

b) Derive an expression for the acceptance probability α.

c) Explain how the inverse CDF method can be used to generate random variables from the distribution of X? What are there advantages and disadvantages of this approach?

d) Propose a method which is both accurate and computationally efficient for generating samples from the distribution of X.

Name: ______________________________