EE 641 Midterm Exam

October 18, Fall 2013

Name: Key

Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 108 points.
- You may not use any notes, textbooks, or calculators.
- You are allowed 50 minutes to complete the exam. Hand in the exam after that period **whether or not you have completed it.**

Good luck.

Problem 1. (36pt)

Let $X \sim N(0, R)$ where R is a $p \times p$ symmetric positive-definite matrix with an eigen decomposition of the form $R = E\Lambda E^t$.

a) Calculate the covariance of $\tilde{X} = E^t X$, and show that the components of $\tilde{X}$ are jointly independent Gaussian random variables.

b) Show that if $Y = E\Lambda^{1/2}W$ where $W \sim N(0, I)$, then $Y \sim N(0, R)$.

c) How can this result be of practical value?

a)
$$E[\tilde{X}\tilde{X}^t] = E[E^t X X^t E]$$
$$= E^t E[XX] E$$
$$= E^t E \Lambda E^t E$$
$$= \Lambda$$

b)
$$Y = E\Lambda^{1/2} W$$
$$\mu_Y = E[Y] = E[E\Lambda^{1/2}W] = E\Lambda^{1/2} E[W] = 0$$
$$R_Y = E[E\Lambda^{1/2} W W^t \Lambda^{1/2} E^t]$$
$$= E\Lambda^{1/2} E[WW^t] \Lambda^{1/2} E^t$$
$$= E\Lambda^{1/2} I \Lambda^{1/2} E^t$$
$$= E \Lambda E^t = R$$

Also, Y must be jointly Gaussian because it is a linear transform of a Gaussian random vector W.

Name: ____________________

b) continued...

Another solution

$$A = E\Lambda^{1/2} \qquad Y = AX$$

$$A^{-1} = \Lambda^{-1/2} E$$

$$(A^{-1})^{t} = E^{t}\Lambda^{-1/2}$$

$$p_Y(y) = \frac{1}{|A|} p_W(A^{-1}y) = \frac{1}{2} \exp\left\{-\frac{1}{2}(A^{-1}Y)^{t} I (A^{-1}Y)\right\}$$

$$= \frac{1}{2} \exp\left\{-\frac{1}{2} Y^{t} E \Lambda^{-1/2} I \Lambda^{-1/2} E^{t} Y\right\}$$

$$= \frac{1}{2} \exp\left\{-\frac{1}{2} Y^{t} E \Lambda^{-1} E^{t} Y\right\}$$

$$= \frac{1}{(2\pi)^{p/2}} |R|^{-1/2} \exp\left\{-\frac{1}{2} Y^{t} R^{-1} Y\right\}$$

$$R^{-1} = E\Lambda^{-1}E^{t} \qquad R = E\Lambda E^{t}$$

$$\Rightarrow Y \sim N(0, R)$$

c) ~~Part~~ b) provides a practical method for generating Gaussian random vectors with any desired covariance.

Problem 2. (36pt)
Let X_s be a zero-mean 2-D Gaussian AR process indexed by $s = (s_1, s_2)$, and let the MMSE causal prediction filter be given by

$$h_s = \rho\delta_{s_1-1,s_2} + \rho\delta_{s_1,s_2-1}$$

and the causal prediction variance be given by σ_C^2. Determine (σ_{NC}^2, g_s) the noncausal prediction variance and the noncausal prediction filter.

$$\frac{\sigma_{NC}^2}{1-G(\omega)} = \frac{\sigma_C^2}{|1-H(\omega)|^2} \Rightarrow$$

$$\sigma_{NC}^2 (\delta_n - h(n)) * (\delta_n - h(-n)) = \sigma_C^2 (\delta_n - g(n))$$

$$(\delta_s - h(s)) * (\delta_s - h(-s)) = \frac{\sigma_C^2}{\sigma_{NC}^2} (\delta_s - g(s))$$

$$\begin{matrix} & -\rho \\ -\rho & 1 \end{matrix} \quad \text{auto correlate} \Rightarrow \quad \begin{matrix} 0 & -\rho & \rho^2 \\ -\rho & 1+2\rho^2 & -\rho \\ \rho^2 & -\rho & 0 \end{matrix}$$

$$\sigma_{NC}^2 = \frac{1}{1+2\rho^2}$$

$$\Rightarrow \begin{bmatrix} 0 & \frac{-\rho}{1+2\rho^2} & \frac{\rho^2}{1+2\rho^2} \\ \frac{-\rho}{1+2\rho^2} & 0 & \frac{-\rho}{1+2\rho^2} \\ \frac{\rho^2}{1+2\rho^2} & \frac{-\rho}{1+2\rho^2} & 0 \end{bmatrix} = -g(s)$$

Name: ______________________

$$g(s) = \begin{cases} 0 & s = (0,0) \\ \dfrac{-\rho}{1+2\rho^2} & s = (1,0), (-1,0), (0,1), (0,-1) \\ \dfrac{\rho^2}{1+2\rho^2} & s = (1,-1)\ (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

Problem 3. (36pt)
Show that if X is a zero-mean GMRF with inverse covariance, B, then $B_{s,r} = 0$ if and only if x_s is conditionally independent of X_r given $\{X_i : \text{ for } i \neq s, r\}$.

let w.l.o.g. $B = \begin{bmatrix} \frac{1}{\sigma^2} & \beta^t \\ \beta & C \end{bmatrix}$ $\quad X = \begin{bmatrix} x_s \\ Y \end{bmatrix}$

Then

$$p(x_s | Y) = \frac{1}{z} \exp\left\{-\frac{1}{2} x^t B x\right\}$$

where $z = \int_{\mathbb{R}} p(x_s, y)\, dx_s$

$$x^t B x = \frac{1}{\sigma^2} x_s^2 + 2\beta^t Y x_s + \text{constant}$$

$$= \frac{1}{\sigma^2}\left(x_s - \sigma^2 \beta^t Y\right)^2 + \text{constant}$$

$$p(x_s | Y) = \frac{1}{z} \exp\left\{-\frac{1}{2\sigma^2}(x_s - \sigma^2\beta^t y)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x_s - \sigma^2\beta^t y)^2\right\}$$

$$\sim N\left(\sigma^2\beta^t Y, \sigma^2\right)$$

$$E[x_s | Y] = \sigma^2\beta^t Y \quad \text{Var}[x_s | Y] = \sigma^2$$

Name: ______________________

Since B is positive definite $\Rightarrow \sigma^2 > 0$

so

X_s is conditionally independent of X_i given $\{X_i$ for $i \neq s, r\}$

$\Leftrightarrow$ $p(x_s \mid x_i, i \neq s)$ is not a function of x_r

$\Leftrightarrow$ $\sigma^2 \beta^t y$ is not a function of x_r

$\Leftrightarrow$ $\beta_r = 0$

$\Leftrightarrow$ $B_{sr} = 0 = B_{rs}$