EE 641 Midterm Exam

October 12, Fall 2012

Name: Key

Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 108 points.
- You may not use any notes, textbooks, or calculators.

Good luck.

Problem 1. (36pt)

Let $Y \in \mathbb{R}^M$ and $X \in \mathbb{R}^N$ be zero-mean jointly Gaussian random vectors. Then define the following notation for this problem. Let $p(y,x)$ and $p(y|x)$ be the joint and conditional density of Y given X. Let B be the joint positive-definite precision matrix (i.e., inverse covariance matrix) given by $B^{-1} = \mathbb{E}[ZZ^t]$ where $Z = \begin{bmatrix} Y \\ X \end{bmatrix}$. Furthermore, let C, D, and E be the matrix blocks that form B, so that

$$B = \begin{bmatrix} C & D \\ D^t & E \end{bmatrix}.$$

where $C \in \mathbb{R}^{M\times M}$, $D \in \mathbb{R}^{M\times N}$, and $E \in \mathbb{R}^{N\times N}$. Finally, define the matrix A so that $AX = \mathbb{E}[Y|X]$ and the matrix $\Lambda^{-1} = \mathbb{E}[YY^t|X]$.

a) Write out an expression for $p(y,x)$ in terms of B.

b) Write out an expression for $p(y|x)$ in terms of A and Λ.

c) Derive an expression for Λ the in terms of C, D, and E.

d) Derive an expression for A the in terms of C, D, and E.

a) $p(y,x) = \frac{1}{(2\pi)^{\frac{M+N}{2}}} |B|^{1/2} \exp\left\{ -\frac{1}{2} \begin{bmatrix} Y \\ X \end{bmatrix}^t B \begin{bmatrix} Y \\ X \end{bmatrix} \right\}$

b) $p(y|x) = \frac{1}{(2\pi)^{M/2}} |\Lambda|^{1/2} \exp\left\{ -\frac{1}{2} (Y-AX)^t \Lambda (Y-AX) \right\}$

c) $Y^t C Y + 2Y^t D X + X^t E X = Y^t \Lambda Y - 2Y^t \Lambda A X + X^t A^t \Lambda A X$

$\Lambda = C$

d) $-\Lambda A = D$

$A = -\Lambda^{-1} D$

$A = -C^{-1} D$

Problem 2. (36pt)

Let X_n be a 1-D zero-mean Gaussian AR process with MMSE causal prediction filter given by $h_n = \rho\delta_{n-1}$ and causal prediction variance σ_C^2.

a) Calculate, $S_X(\omega)$, the power spectral density of the random process.

b) Calculate, $R_X(n)$, the time autocorrelation of the random process.

c) Calculate (σ_{NC}^2, g_n) the noncausal prediction variance and the noncausal prediction filter for the equivalent GMRF. Hint: Remember that

$$\sigma_{NC}^2(\delta_n - h_n) * (\delta_n - h_{-n}) = \sigma_C^2(\delta_n - g_n) .$$

a) $$S_X(\omega) = \frac{\sigma_c^2}{|1-H(\omega)|^2} = \frac{\sigma_c^2}{|1-\rho e^{-j\omega}|^2}$$

$$= \frac{\sigma_c^2}{1+\rho^2 - 2\rho\cos(\omega)}$$

$$= \frac{\sigma_c^2}{(1+\rho^2)\left(1 - \frac{2\rho}{1+\rho^2}\cos(\omega)\right)}$$

b) $$R_X(n) = \sigma_c^2 \left(\rho^n u(n)\right) * \left(\rho^{-n} u(-n)\right)$$

$$= \sigma_c^2 \rho^{-|n|} \frac{1}{1-\rho^2}$$

c) $$(\delta_n - h_n) * (\delta_n - h_{-n})$$

$$= (1+\rho^2)\delta_n + \rho(\delta_{n-1} + \delta_{n+1})$$

$$\sigma_{NC}^2 = \frac{\sigma_c^2}{1+\rho^2} \qquad g_n = \begin{cases} \frac{\rho}{1+\rho^2} & \text{if } n = \pm 1 \\ 0 & \text{o.w.} \end{cases}$$

Problem 3. (36pt)
Consider the optimization problem

$$\hat{x} = \arg\min_{x \in \mathbb{R}^N} \left\{ ||y - Ax||_\Lambda^2 + x^t B x \right\}$$

where A is a nonsingular $N \times N$ matrix, B is a positive-definite $N \times N$ matrix, and Λ is a diagonal and positive-definite matrix..

a) Derive a closed form expression for the solution.

b) Calculate an expression for the gradient descent update using step size $\mu \geq 0$.

c) Calculate an expression for the coordinate descent update.

a) $\nabla_X \left\{ (Y - AX)^t \Lambda (Y - AX) + X^t B X \right\} = 0$

$-2(Y - AX)^t \Lambda A + 2X^t B = 0$

$-A^t \Lambda (Y - AX) + BX = 0$

$-A^t \Lambda Y + A^t \Lambda A X + BX = 0$

$(A^t \Lambda A x + B) X = A^t \Lambda Y$

$X = (A^t \Lambda A + B)^{-1} A^t \Lambda Y$

b) $X^{(K+1)} = X^{(K)} + \alpha \left(A^t \Lambda (Y - AX) - BX \right)$

Name: ______________________

c) $\cancel{x_i^{(k+1)} \leftarrow x_i +}$

$$f(\alpha) = (e - \alpha A_{*i})^* \Lambda (e - \alpha A_{*i}) + (x + \alpha \varepsilon_i)^* B (x - \alpha \varepsilon_i)$$

$$f(\alpha) = c + \theta_1 \alpha + \tfrac{1}{2} \theta_2 \alpha^2$$

$$\theta_1 = \left.\frac{d f(\alpha)}{d\alpha}\right|_{\alpha=0} = -2e^* \Lambda A_{*i} + 2x^* B_{*i}$$

$$\theta_2 = \left.\frac{d^2 f(\alpha)}{d\alpha}\right|_{\alpha=0} = 2A_{*i}^* \Lambda A_{*i} + 2B_{ii}$$

$$\frac{d f(\alpha)}{d\alpha} = 0 = \theta_1 + \theta_2 \alpha \qquad \alpha = \frac{-\theta_1}{\theta_2}$$

$$\alpha = \frac{e^* \Lambda A_{*i} - x^* B_{*i}}{A_{*i}^* \Lambda A_{*i} + B_{ii}}$$

$$x_i \leftarrow x_i + \frac{e^* \Lambda A_{*i} - x^* B_{*i}}{A_{*i}^* \Lambda A_{*i} + B_{ii}}$$