EE 641 Final Exam

Fall 2012

Name: Bouman

Instructions

- This exam contains 4 problems worth a total of 100 points.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (25pt)
Consider the function

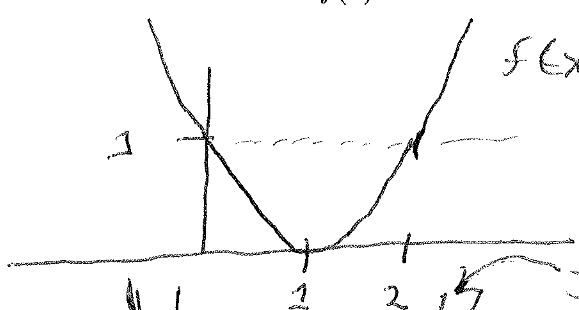
$$f(x) = |x - x_r|^{1.1} \ ,$$

for $x \in \Re$.
a) Sketch a plot of $f(x)$ when $x_r = 1$.
b) Sketch a good substitute function, $f(x; x')$, for $x_r = 1$ and $x' = 2$.
c) Determine a general expression for the substitute function $f(x; x')$ that works for any value of x_r and x'.
d) Assuming the objective is to minimize the expression

$$f(x) = \sum_{r=1}^{P} |x - x_r|^{1.1} \ ,$$

for $x \in \Re$, specify an iterative algorithm in terms of the substitute function $f(x; x')$ for the minimization of the function $f(x)$.

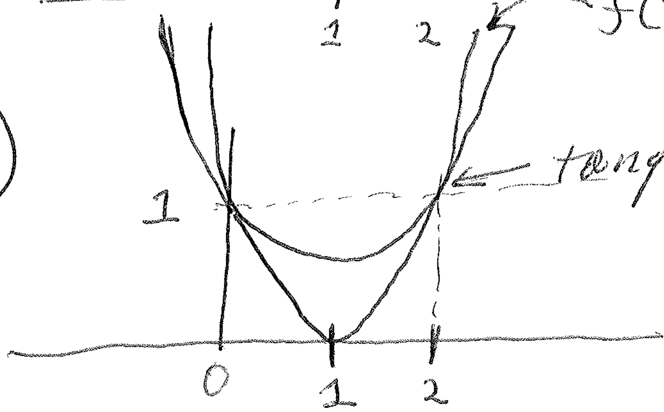
a)



$f(x) = |x - 1|^{1.1}$

$f(x;2)$

b)



tangent at $x = 2$ and 0

c) If $\rho(x) \Leftarrow$ symmetric $\quad \rho(x) = |x|^{1.1}$

$\rho(x; x') = \frac{a}{2}(x)^2$ where $a = \frac{\rho'(x')}{x'}$

$a = \frac{(1.1)|x'|^{0.1}}{|x'|}$

$= \frac{(1.1)}{|x'|^{0.9}}$

So $f(x; x') = \frac{a}{2}(x - x_n)^2$

$$= \frac{(1.1)}{2} \frac{(x - x_n)^2}{|x' - x_n|^{0.9}}$$

d)

$$f(x; x') = \sum_{n=1}^{P} \left(\frac{1.1}{2}\right) \frac{(x - x_n)^2}{|x' - x_n|^{0.9}}$$

then for $n = 1$ to N {

$x' \leftarrow \arg\min_{x} f(x; x')$

}

Problem 2.(25pt)

Let X_n be a sequence of i.i.d. random variables for $n = 1, \cdots, N$, and let Y_n be a sequence of random variables that are conditionally independent and exponentially distributed with mean μ_i for $i = 0, \cdots, M-1$ given the corresponding values of X_n. More specifically, the distributions of X and Y are given by

$$\begin{aligned} P\{X_n = i\} &= \pi_i \\ p(y_n|x_n) &= \frac{1}{\mu_{x_n}} e^{-y_n/\mu_{x_n}} u(y_n) \end{aligned}$$

where $i \in \{0, \cdots, M-1\}$ and $u(\cdot)$ is the unit step function. Furthermore, let $\theta = [\pi_0, \mu_0, \cdots, \pi_{M-1}, \mu_{M-1}]$ be the parameter vector of the distribution.

a) Derive a closed form expression for the ML estimate of θ given both X and Y (i.e., the complete data).

b) Find the natural sufficient statistics for the exponential distribution of (X, Y).

c) Specify the EM algorithm for computing the ML estimate of θ from Y.

~~a~~b) $N_i = \sum_{n=1}^{N} \delta(x_n - i) \qquad N = \sum_{i=1}^{M-1} N_i$

$S_i = \sum_{n=1}^{N} y_n \delta(x_n - i)$

a) $\hat{\pi}_i = \frac{N_i}{N}$

$\hat{\mu}_i = \frac{S_i}{N_i}$

c) E-step $\quad \bar{N}_i \leftarrow \sum_{n=1}^{N} P\{X_n = i | Y, \theta\}$

$\bar{S}_i \leftarrow \sum_{n=1}^{N} y_n P\{X_n = i | Y, \theta\}$

M-step

$$\hat{\pi}_i \leftarrow \frac{\overline{n_i}}{N}$$

$$\hat{\mu}_i \leftarrow \frac{\overline{S_i}}{\overline{n_i}}$$

Problem 3.(25pt)
Consider a Markov chain, $\{X_n\}_{n=0}^{\infty}$, with the transition probabilities,

$$P_{i,j} = \begin{cases} \lambda & \text{if } j = i+1 \\ \mu & \text{if } j = i-1 \text{ and } i > 0 \\ 1-\lambda-\mu & \text{if } j = i \text{ and } i > 0 \\ 1-\lambda & \text{if } j = i \text{ and } i = 0 \\ 0 & \text{otherwise} \end{cases},$$

where $\rho = \frac{\lambda}{\mu}$ is in the range $0 < \rho < 1$.

a) Find a stationary distribution, π_k, and prove that it solves the full balance equations for this Markov chain.

b) Show that the Markov chain is not periodic and is irreducible.

c) Is the Markov chain ergodic for $\rho \in (0,1)$? Is it ergodic for $\rho > 1$? Explain your reasoning.

a) $\pi_i P_{ij} = \pi_j P_{ji}$

$\pi_i \lambda = \pi_{i+1} \mu$

$\frac{\pi_{i+1}}{\pi_i} = \frac{\lambda}{\mu} = \rho \Rightarrow \pi_i = \rho^i \pi_0 \qquad \pi_0 \sum_{i=0}^{\infty} \rho^i = 1$

$\pi_0 = 1-\rho$

$\pi_i = \rho^i(1-\rho)$

b) If $i > j$ then $\forall n > i-j$ we have that

$[P^n]_{ji} \geq (\lambda)^{i-j}(1-\lambda-\mu)^{n-(i-j)} > 0$

If $i < j$ then $\forall n > j-i$,

$[P^n]_{ji} > (\mu)^{j-i}(1-\lambda-\mu)^{n-(j-i)} > 0$

If $i = j$, $[P^n]_{ij} \geq (1-\lambda-\mu)^n > 0$

$\Rightarrow [P^n]_{ij} > 0 \Rightarrow$ irreducible
$\Rightarrow$ aperiodic

c) Yes, it is ergodic for $\rho \in (0,1)$ because there is a solution to the FBE given by

$$\pi_i = \frac{\rho^i}{1-\rho} \quad \text{for } i \geq 0$$

No, it is not ergodic for $\rho > 1$ because $\sum_{i=0}^{\infty} \rho^i = \infty$.

Intuitively, the $\lim_{n \to \infty} X_n = \infty$.

So the Markov chain's state tends toward infinity.

Problem 4.(25pt)

a) Prove that a 1-D Markov chain of order $P = 1$ is also an MRF of order $P = 1$.

b) Prove that a 1-D MRF of order $P = 1$ is also a Markov chain of order $P = 1$.

a) $$p(x) = p(x_0) \prod_{n=1}^{N} p(x_n | x_{n-1})$$

$$p(x) = \frac{1}{z} \exp\left\{ -V_0(x_0) - \sum_{n=1}^{N} V_n(x_n, x_{n-1}) \right\}$$

where

$$V_0(x_0) = -\log p_0(x_0)$$

$$V_n(x_n, x_{n-1}) = -\log p_n(x_n | x_{n-1})$$

by the Hammersly Clifford theorem $\Rightarrow$ MRF

b) General form of 1-D order $P=1$ MRF is

$$p(x) = \frac{1}{Z} \exp\left\{ -V_0(x_0) - \sum_{n=1}^{N} V_n(x_n, x_{n-1}) \right\}$$

$$p(x_0, \dots, x_{N-1}) = \int p(x_0, \dots, x_N)\, dx_N$$

$$= \frac{1}{z} \exp\left\{ -V_0(x_0) - \sum_{n=1}^{N-1} V_n(x_n, x_{n-1}) \right\} \cdot \underbrace{\int_{\mathbb{R}} \exp\{-V_N(x_N, x_{N-1})\}\, dx_N}_{V_N'(}$$

Define

$$V'_{N-1}(x_{N-1}, x_{N-2})$$
$$= V_{N-1}(x_{N-1}, x_{N-2}) - \log \int_{\mathbb{R}} \exp\{-V_N(x_N, x_{N-1})\} \, dx_N$$

then

$$p(x_0, \dots, x_{N-1}) =$$
$$\frac{1}{Z} \exp\left\{-V_0(x_0) - \sum_{n=1}^{N-2} V_n(x_n, x_{n-1}) + V_{N-1}(x_{N-1}, x_{N-2})\right\}$$

So we have that

$$p(x_N | x_n \; n < N) = \frac{p(x_0, \dots, x_N)}{p(x_0, \dots, x_{N-1})}$$
$$= \frac{\exp\{-V_N(x_N, x_{N-1}) - V_{N-1}(x_{N-1}, x_{N-2})\}}{\exp\{-V'_{N-1}(x_{N-1}, x_{N-2})\}}$$
$$= f(x_N, x_{N-1}) = p(x_N | x_{N-1})$$

By induction, this can be repeated for all $N \Rightarrow X_n$ is a Markov chain.