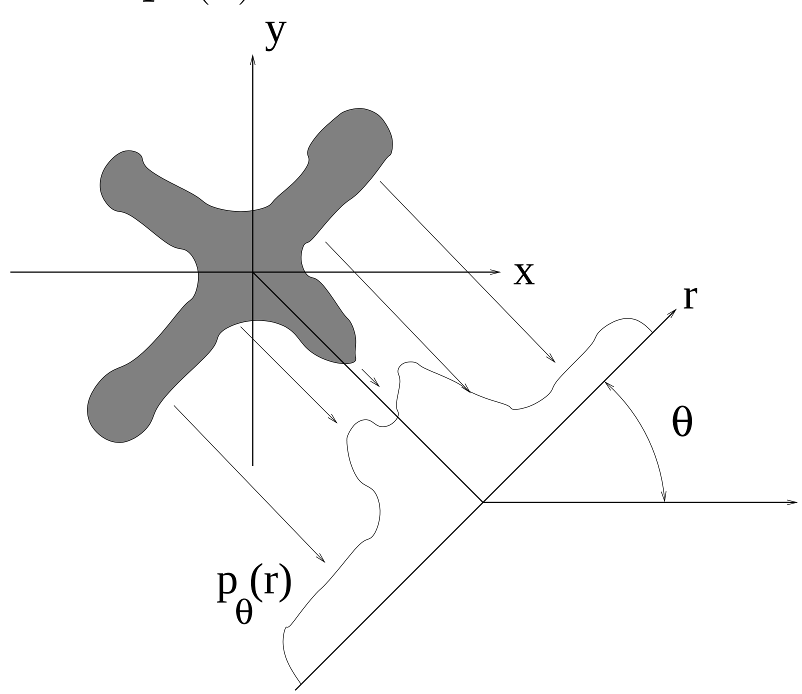
Application of Inverse Methods to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

Forward Projection

- Typical tomographic imaging senerio:
  - Projections collected at every angle θ and displacement r.
  - Forward projections $p_\theta(r)$ are known as a Radon transform.

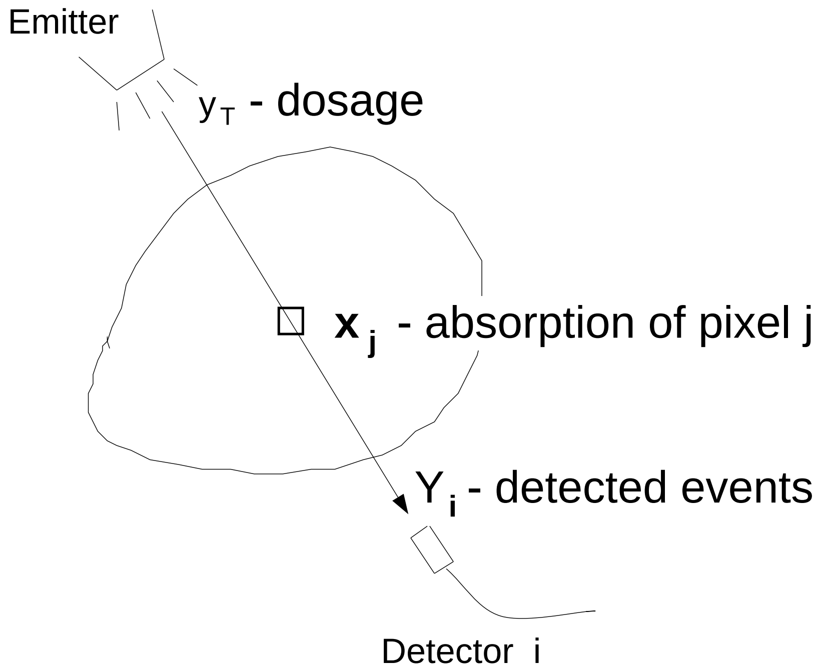


- Objective: reverse this process to form the original image $f(x, y)$.
  - Fourier Slice Theorem is the basis of inverse
  - Inverse can be computed using convolution back projection (CBP)

Advantages of Iterative/Statistical Reconstruction

- Low signal-to-noise data
  - Data may vary with projection (dense objects, noisy detectors, etc.)
  - FBP treats all projections equally
- Missing projections
  - Dense objects may make some views impossible.
  - Helical scanners do not take every view at each position
- Complex geometries
  - Projections may be taken in fan-beam and cone-beam geometries
- Non-Gaussian prior modeling
  - Non-Gaussian models may be particularly appropriate for object cross-sections

Transmission Tomography



Y_T - Dosage emitted from source (not random)

X_j - j^{th} pixel

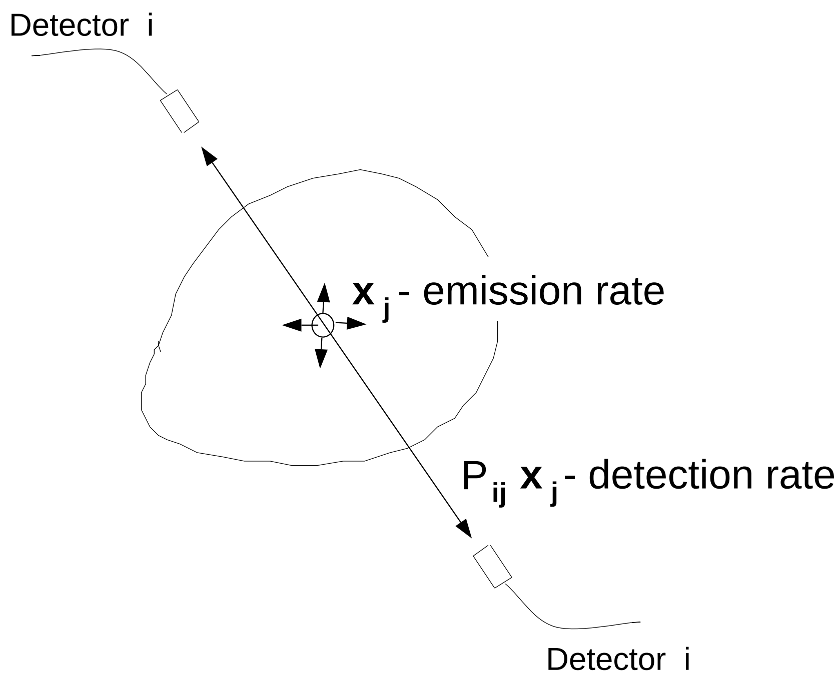
Y_i - Energy measured by i^{th} detector

P_{ij} - Contribution of j^{th} pixel to i^{th} detector

- Typical assumptions
  - Y_i are i.i.d. and Poisson
  - $E[Y_i|X] = Y_T \exp\{\Sigma_j P_{i,j} X_j\}$

- Includes computed tomography (CT), scanning electron microscope (SEM)

Emission Tomography



X_j - Emission rate from j^{th} pixel

Y_i - Energy measured by i^{th} detector *pair*

P_{ij} - Contribution of j^{th} pixel to i^{th} detector

- Typical assumptions
  - Y_i are i.i.d. and Poisson
  - $E[Y_i|X] = \Sigma_j P_{i,j} X_j$

- Includes positron emission tomography (PET), and single photon emission tomography (SPECT)

Statistical Data Model[3]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\Sigma_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$
\begin{aligned}
\hat{x} &= \arg\max_{x} p(x|y) \\
&= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\, \rho(x_k - x_j) \right\}
\end{aligned}
$$

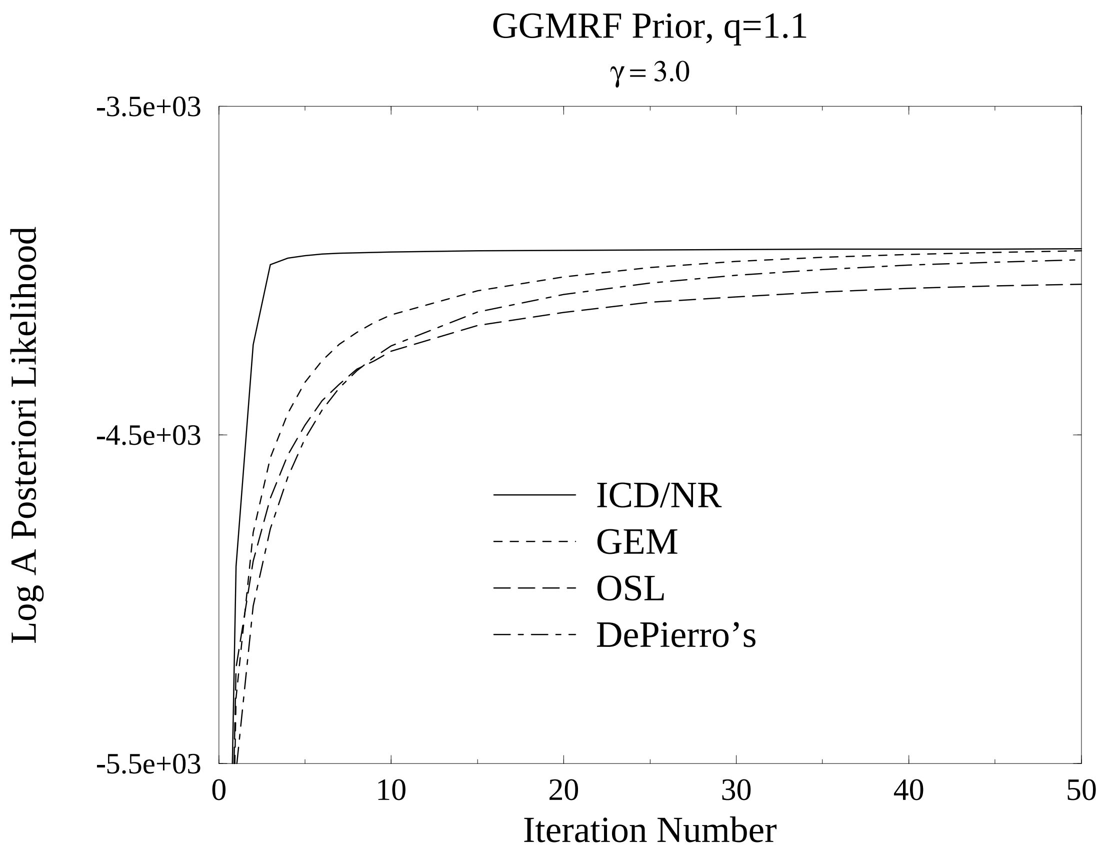
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[12, 10]
  - MAP reconstruction[8, 7, 9]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[6]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[5]
  - ICM iterations (also known as ICD and Gauss-Seidel)[3]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

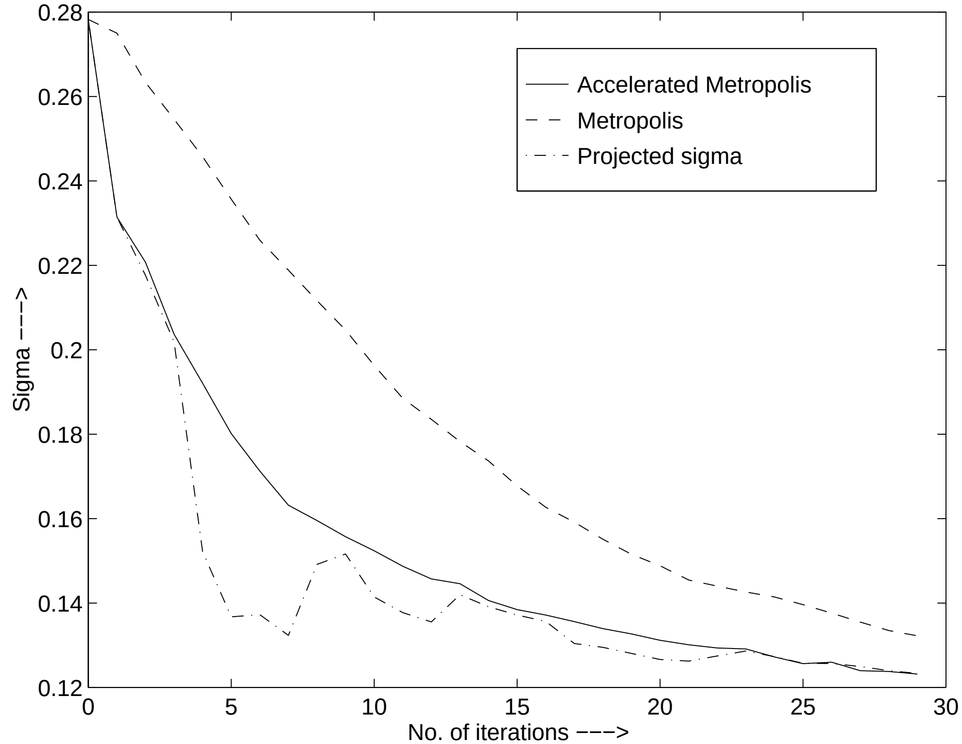
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{\frac{1}{p\sigma^p} U(x)\right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned}\sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{\log p(X|\sigma)|Y = y, \sigma^{(k)}\right\} \\ &= \left(E\left\{\frac{1}{N} U(X)|Y = y, \sigma^{(k)}\right\}\right)^{1/p}\end{aligned}$$

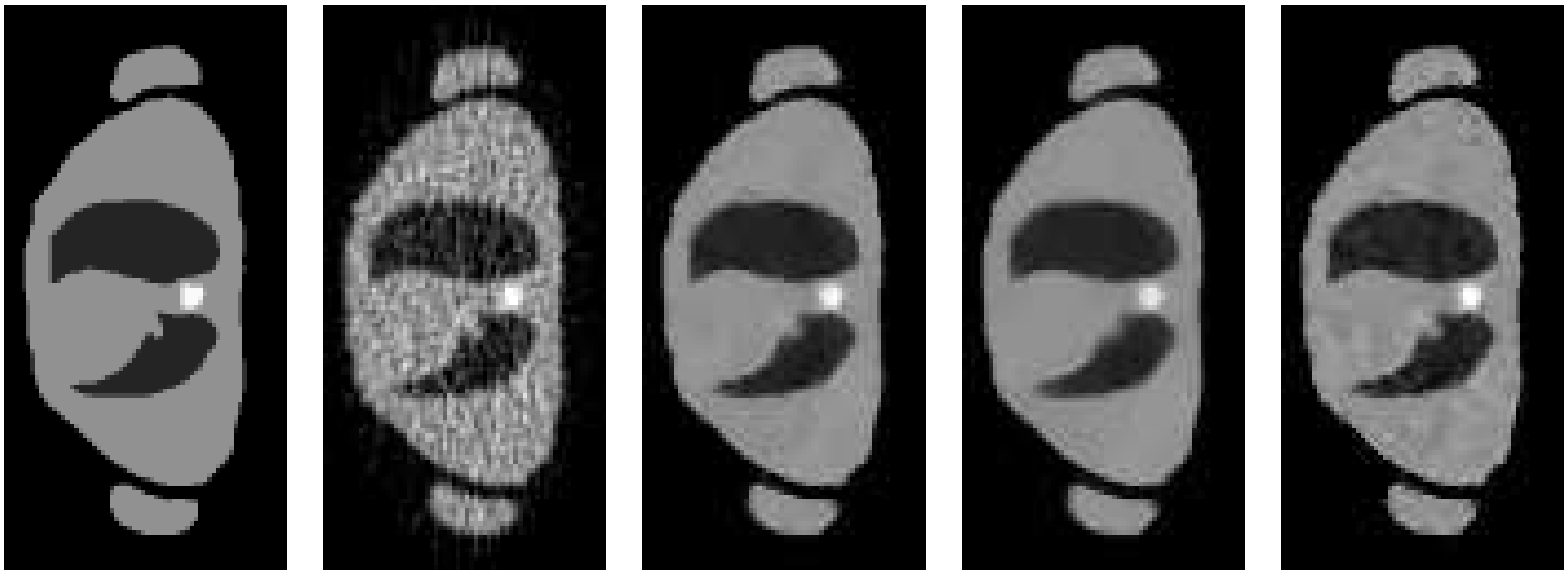
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

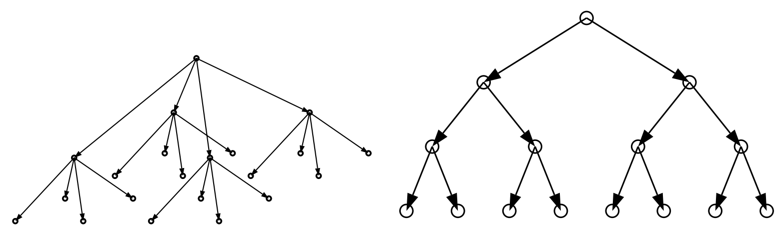


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[2, 1, 11]
  - Discrete models[4, 11]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

References

[1] M. Basseville, A. Benveniste, K. C. Chou, S. A. Golden, R. Nikoukhah, and A. S. Willsky. Modeling and estimation of multiresolution stochastic processes. *IEEE Trans. on Information Theory*, 38(2):766–784, March 1992.

[2] A. Benveniste, R. Nikoukhah, and A. Willsky. Multiscale system theory. In *Proceedings of the 29th Conference on Decision and Control*, volume 4, pages 2484–2489, Honolulu, Hawaii, December 5-7 1990.

[3] C. A. Bouman and K. Sauer. A unified approach to statistical tomography using coordinate descent optimization. *IEEE Trans. on Image Processing*, 5(3):480–492, March 1996.

[4] C. A. Bouman and M. Shapiro. A multiscale random field model for Bayesian image segmentation. *IEEE Trans. on Image Processing*, 3(2):162–177, March 1994.

[5] E. *Ü*. Mumcuoğlu, R. Leahy, S. R. Cherry, and Z. Zhou. Fast gradient-based methods for Bayesian reconstruction of transmission and emission pet images. *IEEE Trans. on Medical Imaging*, 13(4):687–701, December 1994.

[6] J. Fessler and A. Hero. Space-alternating generalized expectation-maximization algorithms. *IEEE Trans. on Acoustics Speech and Signal Processing*, 42(10):2664–2677, October 1994.

[7] P. J. Green. Bayesian reconstruction from emission tomography data using a modified EM algorithm. *IEEE Trans. on Medical Imaging*, 9(1):84–93, March 1990.

[8] T. Hebert and R. Leahy. A generalized EM algorithm for 3-D Bayesian reconstruction from Poisson data using Gibbs priors. *IEEE Trans. on Medical Imaging*, 8(2):194–202, June 1989.

[9] G. T. Herman, A. R. De Pierro, and N. Gai. On methods for maximum a posteriori image reconstruction with normal prior. *J. Visual Comm. Image Rep.*, 3(4):316–324, December 1992.

[10] K. Lange. Convergence of EM image reconstruction algorithms with Gibbs smoothing. *IEEE Trans. on Medical Imaging*, 9(4):439–446, December 1990.

[11] M. R. Luettgen, W. C. Karl, and A. S. Willsky. Efficient multiscale regularization with applications to the computation of optical flow. *IEEE Trans. on Image Processing*, 3(1):41–64, January 1994.

[12] L. Shepp and Y. Vardi. Maximum likelihood reconstruction for emission tomography. *IEEE Trans. on Medical Imaging*, MI-1(2):113–122, October 1982.