Simulation

- Topics to be covered:
  - Gibbs sampler
  - Metropolis sampler
  - Hastings-Metropolis sampler

Generating Samples from a Gibbs Distribution

- How do we generate a random variable X with a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\left\{-U(x)\right\}$$

- Generally, this problem is difficult.
- Markov Chains can be generated sequentially
- Non-causal structure of MRF's makes simulation difficult.

Gibbs Sampler[?]

- Replace each point with a sample from its conditional distribution

$$p(x_s|x_i^k\ i \neq s) = p(x_s|x_{\partial s})$$

- Scan through all the points in the image.
- Advantage
  - Eliminates need for rejections $\Rightarrow$ faster convergence
- Disadvantage
  - Generating samples from $p(x_s|x_{\partial s})$ can be difficult.

Gibbs Sampler Algorithm

Gibbs Sampler Algorithm:

1. Set $N = \#$ of pixels
2. Order the N pixels as $N = s(0), \cdots, s(N-1)$
3. Repeat for $k = 0$ to ∞
   (a) Form $X^{(k+1)}$ from $X^{(k)}$ via

$$X_r^{(k+1)} = \begin{cases} W & \text{if } r = s(k) \\ X_r^{(k)} & \text{if } r \neq s(k) \end{cases}$$

where $W \sim p\left(x_{s(k)} \middle| X_{\partial s(k)}^{(k)}\right)$

The Metropolis Sampler[?, ?]

- How do we generate a sample from a Gibbs distribution?

$$p(x) = \frac{1}{Z} \exp\{-U(x)\}$$

- Start with the sample x^k, and generate a new sample W with probability $q(w|x^k)$.

  Note: $q(w|x^k)$ must be symmetric.

$$q(w|x^k) = q(x^k|w)$$

- Compute $\Delta E(W) = U(W) - U(x^k)$, then do the following:

  If $\Delta E(W) < 0$

  - Accept: $X^{k+1} = W$

  If $\Delta E(W) \geq 0$

  - Accept: $X^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
  - Reject: $X^{k+1} = x^k$ with probability $1 - \exp\{-\Delta E(W)\}$

Ergodic Behavior of Metropolis Sampler

- The sequence of random fields, X^k, form a Markov chain.
- Let $p(x^{k+1}|x^k)$ be the transition probabilities of the Markov chain.
- Then X^k is reversible

$$p(x^{k+1}|x^k)\exp\{-U(x^k)\} = \exp\{-U(x^{k+1})\}p(x^k|x^{k+1})$$

- Therefore, if the Markov chain is irreducible, then

$$\lim_{k\to\infty} P\{X^k = x\} = \frac{1}{Z}\exp\{-U(x)\}$$

- If every state can be reached, then as $k \to \infty$, X^k will be a sample from the Gibbs distribution.

Example Metropolis Sampler for Ising Model

	0	
1	x_s	0
	0	

- Assume $x_s^k = 0$.
- Generate a binary R.V., W, such that $P\{W = 0\} = 0.5$.

$$\begin{aligned} \Delta E(W) &= U(W) - U(x_s^k) \\ &= \begin{cases} 0 & \text{if } W = 0 \\ 2\beta & \text{if } W = 1 \end{cases} \end{aligned}$$

If $\Delta E(W) < 0$

- Accept $X_s^{k+1} = W$

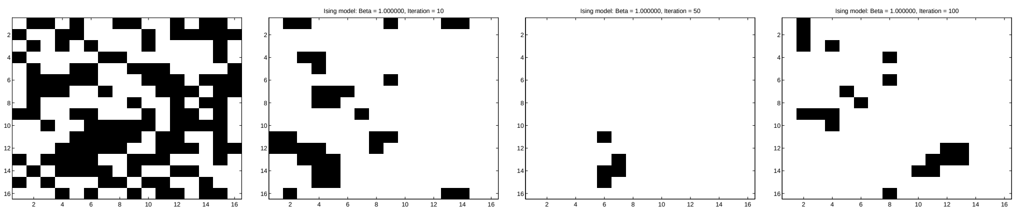
If $\Delta E(W) \geq 0$

- Accept: $X_s^{k+1} = W$ with probability $\exp\{-\Delta E(W)\}$
- Reject: $X_s^{k+1} = x_s^k$ with probability $1 - \exp\{-\Delta E(W)\}$

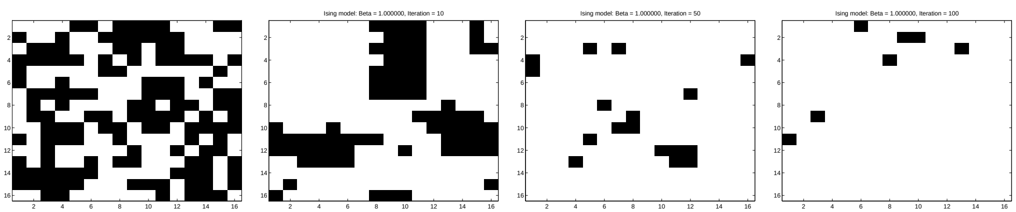
- Repeat this procedure for each pixel.
- **Warning:** for $\beta > \beta_c$ convergence can be extremely slow!

Example Simulation for Ising Model($\beta = 1.0$)

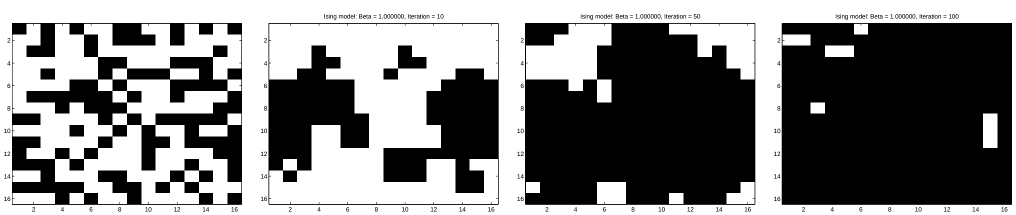
- Test 1



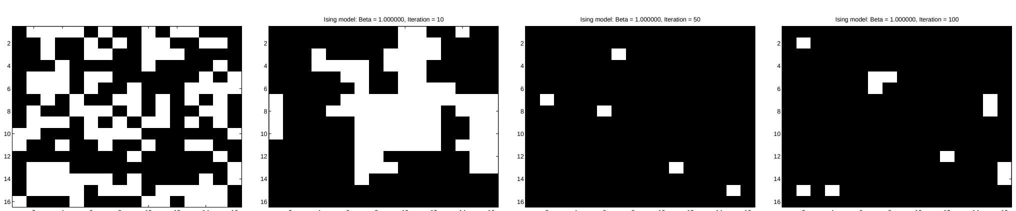
- Test 2



- Test 3



- Test 4



Uniform Random 10 Iterations 50 Iterations 100 Iterations

Advantages and Disadvantages of Metropolis Sampler

- Advantages
  - Can be implemented whenever ΔE is easy to compute.
  - Has guaranteed geometric convergence.
- Disadvantages
  - Can be slow if there are many rejections.
  - Is constrained to use a symmetric transition function $q(x^{k+1}|x^k)$.

Hastings-Metropolis Sampler[?, ?]

- Hastings and Peskun generalized the Metropolis sampler for transition functions $q(w|x^k)$ which are not symmetric.
- The acceptance probability is then

$$\alpha(x_s^k, w) = \min\left\{1, \frac{q(x^k|w)}{q(w|x^k)} \exp\{-\Delta E(w)\}\right\}$$

- Special cases

$$q(w|x^k) = q(x^k|z) \Rightarrow \text{conventional Metropolis}$$
$$q(w_s|x^k) = p(x_s^k|x_{\partial s}^k)\big|_{x_s^k = w_s} \Rightarrow \text{Gibbs sampler}$$

- Advantage
  – Transition function may be chosen to minimize rejections[**?**]

Parameter Estimation for Discrete State MRF's

- Topics to be covered:
  - Why is it difficult?
  - Coding/maximum pseudolikehood
  - Least squares

Why is Parameter Estimation Difficult?

- Consider the ML estimate of β for an Ising model.
- Remember that

$$t_1(x) = (\#\text{ horz. and vert. neighbors of different value.})$$

- Then the ML estimate of β is

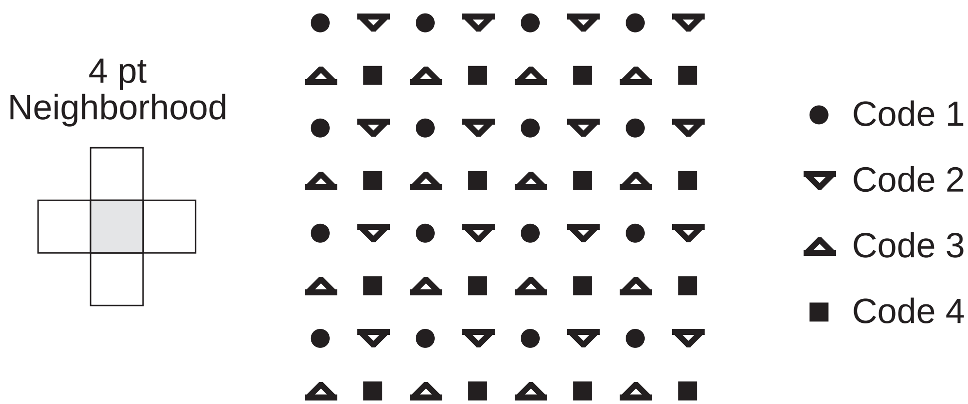
$$\begin{aligned}\hat{\beta} &= \arg\max_{\beta}\left\{\frac{1}{Z(\beta)}\exp\{-\beta t_1(x)\}\right\} \\ &= \arg\max_{\beta}\{-\beta t_1(x) - \log Z(\beta)\}\end{aligned}$$

- However, $\log Z(\beta)$ has an intractable form

$$\log Z(\beta) = \log \sum_x \exp\{-\beta t_1(x)\}$$

- Partition function can not be computed.

Coding Method/Maximum Pseudolikelihood[?, ?]



- Assume a 4 point neighborhood
- Separate points into four groups or codes.
- Group (code) contains points which are conditionally independent given the other groups (codes).

$$\hat{\beta} = \arg\max_{\beta} \prod_{s \in \text{Code}_k} p(x_s | x_{\partial s})$$

- This is tractable (but not necessarily easy) to compute

Least Squares Parameter Estimation[?]

- It can be shown that for an Ising model

$$\log \frac{P\{X_s = 1|x_{\partial s}\}}{P\{X_s = 0|x_{\partial s}\}} = -\beta \left(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s})\right)$$

- For each unique set of neighboring pixel values, $x_{\partial s}$, we may compute
  - The observed rate of $\log \frac{P\{X_s=1|x_{\partial s}\}}{P\{X_s=0|x_{\partial s}\}}$
  - The value of $(V_1(1|x_{\partial s}) - V_1(0|x_{\partial s}))$
  - This produces a set of over-determined linear equations which can be solved for β.
- This least squares method is easily implemented.

Theoretical Results in Parameter Estimation for MRF's

- Inconsistency of ML estimate for Ising model[**?**, **?**]
  - Caused by critical temperature behavior.
  - Single sample of Ising model cannot distinguish between high β with mean 1/2, and low β with large mean.
  - Not identifiable
- Consistency of maximum pseudolikelihood estimate[**?**]
  - Requires an identifiable parameterization.

References