EE 641 Midterm Exam

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Instructions

The following is an in-class closed-book exam.

- This exam contains 3 problems worth a total of 108 points.
- You may not use any notes, textbooks, or calculators.

Good luck.

Problem 1. (36pt)
Consider the p dimensional random vector, X, with distribution

$$p(x) = \frac{1}{z}\exp\left\{-\frac{1}{2}x^t B x\right\}$$

a) Give an expression for the normalizing constant z.
b) Assuming that X is a Markov random field with neighborhood system ∂s for $s \in \{1, \cdots, p\}$, then specify which values of B must be zero.
c) Compute an expression for the conditional density $p(x_s|x_r$ for $r \neq s)$.
d) Compute an expression for the marginal density $p(x_s)$.

a) $$z = (2\pi)^{p/2}\,|B|^{-1/2}$$

b) $$\forall s, r \text{ s.t. } r \notin \partial s \Rightarrow B_{r,s} = B_{s,r} = 0$$

c) $$p(x_s \mid x_r\ r \neq s) = p(x_s \mid x_{\partial s})$$

$$= \frac{1}{z}\exp\left\{-\frac{1}{2\sigma^2}(x_s-\mu)^2\right\}$$

$$\frac{\partial}{\partial x_s}\log p(x_s \mid x_{\partial s}) = \frac{\partial}{\partial x_s}\log p(x)$$

$$-\frac{(x_s-\mu)}{\sigma^2} = -\sum_r x_r B_{rs} = -\sum_{r\in\partial s\cup\{s\}} B_{sr}x_r$$

$$= -\left(B_{ss}x_s + \sum_{r\in\partial s} B_{sr}x_r\right)$$

$$= -B_{ss}\left(x_s + \sum_{r\in\partial s}\frac{B_{sr}}{B_{ss}}x_r\right)$$

$$\sigma^2 = \frac{1}{B_{ss}} \qquad \mu = -\sum_{r\in\partial s}\frac{B_{sr}}{B_{ss}}x_r$$

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d) $E[X X^{t}] = R = B^{-1}$

$$E[x_s^2] = R_{ss} = (B^{-1})_{s,s}$$

$$p(x_s) = \frac{1}{Z} \exp\left\{ - \frac{1}{2(B^{-1})_{ss}} x_s^2 \right\}$$

Problem 2. (36pt)

Consider the function

$$f(x) = |x - x_r|^{1.1} ,$$

for $x \in \Re$.

a) Sketch a plot of $f(x)$ when $x_r = 1$.

b) Sketch a good substitute function, $f(x; x')$, for $x_r = 1$ and $x' = 2$.

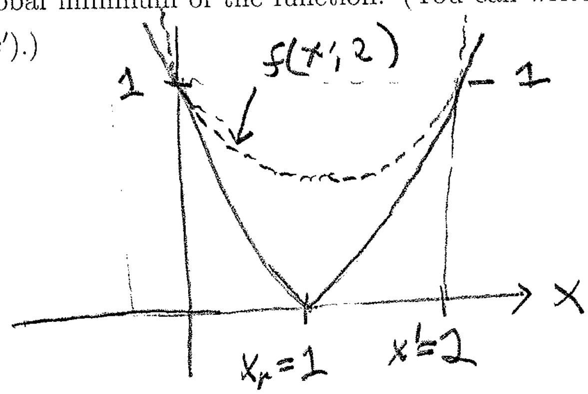
c) Determine a general expression for the substitute function $f(x; x')$ that works for any value of x_r and x'.

d) Assuming the objective is to minimize the expression

$$f(x) = \sum_{r \in \partial s} |x - x_r|^{1.1} ,$$

for $x \in \Re$, specify an iterative algorithm in terms of the substitute function $f(x; x')$ that will converge to the global minimum of the function. (You can write your solution in terms of the function $f(x; x')$.)

a)



b) see sketch above.

c) $$f(x;x') = \frac{a}{2}(x - X_r)^2 + c$$

$$a = f'(2)$$

$$\left.\frac{df(x;x')}{dx}\right|_{x=x'} = \left.\frac{d f(x)}{dx}\right|_{x=x'}$$

$$a(x' - x_r) = f'(x') = (1.1)(x' - x_r)^{0.1} \cdot \mathrm{sign}(x' - x_r)$$

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$$a = (1.1)\ \frac{|x' - x_n|^{0.1}\operatorname{sign}(x' - x_n)}{(x' - x_n)}$$

$$= (1.1)\ |x' - x_n|^{-0.9}$$

$$f(x; x') = (1.1)\ |x' - x_n|^{-0.9}(x - x_n)^2$$

$+ C$ ← doesn't matter

d)

$$x \leftarrow \arg\min_x \sum_{n \in \partial s} (1.1)\ |x' - x_n|^{-0.9}(x - x_n)^2$$

$$x \leftarrow \frac{\sum_{n \in \partial s} (1.1)\,|x' - x_n|^{-0.9}\ x_n}{\sum_{n \in \partial s} (1.1)\,|x' - x_n|^{-0.9}}$$

Problem 3. (36pt)
Let $\{X_n\}_{n=1}^N$ be a i.i.d. random varibles with distribution

$$P\{X_n = m\} = \pi_m ,$$

where $\sum_{m=0}^{M-1} \pi_m = 1$. Also, let Y_n be conditionally independent random variables given X_n, with exponential conditional distribution

$$p(y_n|x_n = m) = \frac{1}{\mu_m} \exp\left\{-\frac{y_n}{\mu_m}\right\} u(y_n) ,$$

where $u(y_n)$ is 1 for $y_n \geq 0$ and 0 otherwise.

a) Write out the density function for the vector Y.

b) What are the natural sufficient statistics for the complete data (X, Y)?

c) Give an expression for the ML estimate of the parameter $\theta = (\pi_0, \mu_0, \cdots, \pi_{M-1}, \mu_{M-1})$ given the complete data (X, Y).

d) Give the EM update equations for computing the ML estimate of the parameter $\theta = (\pi_0, \mu_0, \cdots, \pi_{M-1}, \mu_{M-1})$ given the incomplete data Y.

a) $$p(y) = \prod_{n=1}^{N} \left\{ \sum_{m=0}^{M-1} \frac{\pi_m}{\mu_m} \exp\left\{-\frac{y_n}{\mu_m}\right\} u(y_n) \right\}$$

b) $$N_m = \sum_{n=1}^{N} \delta(X_n - m)$$

$$S_m = \sum_{n=1}^{N} Y_n \, \delta(X_n - m)$$

c) $$\pi_m = \frac{N_m}{N}$$

$$\mu_m = \frac{S_m}{N_m}$$

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d) $P\{X_n = m \mid Y\} = f_n(m)$

$$= \frac{p(y_n \mid x_n = m)\,\pi_m}{\sum_{m=0}^{M-1} p(y_n \mid x_n = m)\,\pi_m}$$

$$f_n(m) = \frac{\frac{1}{\mu_m} e^{-\frac{y_n}{\mu_m}} \pi_m}{\sum_{m=0}^{M-1} \frac{1}{\mu_m} e^{-\frac{y_n}{\mu_m}} \pi_m}$$

$$\bar{N}_m = \sum_{n=1}^{N} f_n(m)$$

$$\bar{S}_m = \sum_{n=1}^{N} Y_n f_n(m)$$

$$\pi_m \leftarrow \frac{\bar{N}_m}{N}$$

$$\mu_m \leftarrow \frac{\bar{S}_m}{\bar{N}_m}$$