EE 641 Final Exam

Fall 2010

Name: Key

Instructions

- This exam contains 4 problems worth a total of 154 points.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (36pt)

Consider the p dimensional random vector, X, with distribution

$$p(x) = \frac{1}{z} \exp\left\{ -\frac{1}{2} x^t B x \right\}$$

a) Give an expression for the normalizing constant z.

b) Assuming that X is a Markov random field with neighborhood system ∂s for $s \in \{1, \cdots, p\}$, then specify which values of B must be zero.

c) Compute an expression for the conditional density $p(x_s | x_r$ for $r \neq s)$.

d) Compute an expression for the marginal density $p(x_s)$.

a) $$z = (2\pi)^{p/2} |B|^{-1/2}$$

b) $$\forall s, r \text{ s.t. } r \notin \partial s \Rightarrow B_{r,s} = B_{s,r} = 0$$

c) $$p(x_s | x_r\ r \neq s) = p(x_s | X_{\partial s})$$

$$= \frac{1}{z} \exp\left\{ -\frac{1}{2\sigma^2} (x_s - \mu)^2 \right\}$$

$$\frac{\partial}{\partial x_s} \log p(x_s | X_{\partial s}) = \frac{\partial}{\partial x_s} \log p(x)$$

$$-\frac{(x_s - \mu)}{\sigma^2} = -\sum_r x_r B_{rs} = -\sum_{r \in \partial s \cup \{s\}} B_{sr} x_r$$

$$= -\left(B_{ss} x_s + \sum_{r \in \partial s} B_{sr} x_r \right)$$

$$= -B_{ss} \left(x_s + \sum_{r \in \partial s} \frac{B_{sr}}{B_{ss}} x_r \right)$$

$$\sigma^2 = \frac{1}{B_{ss}} \qquad \mu = -\sum_{r \in \partial s} \frac{B_{s,r}}{B_{ss}} x_r$$

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d) $E[x x^*] = R = B^{-1}$

$$E[x_s^2] = R_{ss} = (B^{-1})_{s,s}$$

$$p(x_s) = \frac{1}{z} \exp\left\{ -\frac{1}{2(B^{-1})_{ss}} x_s^2 \right\}$$

Problem 2.(72pt)
Let X_n for $n = 0$ to $N-1$ be a discrete-valued Markov chain with homogeneous transition probability matrix P and initial probability density given by

$$\pi_i^{(0)} = P\{X_0 = i\} .$$

Furthermore, assume all the entries of P and $\pi^{(0)}$ are strictly positive and that the number of states of the Markov chain are finite.
a) Write the expression for the probability of the sequence $\{X_n\}_{n=0}^{N-1}$.
b) Write an algorithmic recursion for the probability

$$\pi_i^{(n)} = P\{X_n = i\} .$$

(Hint: The recursion should start at $n = 1$ and increment until $n = N-1$.)
c) Show that that the discrete density, $p(x) = P\{X = x\}$, is a Gibbs distribution with neighborhood system given by

$$\partial n = \{n-1, n+1\} \cap \{0, \cdots, N-1\} .$$

d) Use the Hammersley-Clifford Theorem to Prove that X is a Markov random field with neighborhood system ∂n.
e) Does a stationary distribution exist for this Markov chain? If it does exist, specify a set of equations (or a matrix equation) which can be used to determine the stationary distribution? What is the name of this set of equations?

For the following parts, let π be the stationary distribution for the Markov chain, and assume that $\pi^{(0)} = \pi$.
f) Derive an explicit and closed form expression for the transition probabilities of the time-reversed Markov chain.

$$Q_{j,i} = P\{X_n = i | X_{n+1} = j\} .$$

g) Is the time reversed Markov chain homogeneous? Why or why not?
h) Derive conditions for the Markov chain to be reversible in terms of π and P.

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a) $p(x) = \pi^{(0)}_{x_0} \prod_{n=1}^{N-1} P_{x_{n-1} x_n}$

For $n=1$ to $N-1$ {

b) $\pi^{(n)}_i = \sum_{j=1}^{M} \pi^{(n-1)}_j P_{ji}$

}

c) $\log p(x) = \underbrace{\log \pi^{(0)}_{x_0}}_{-V_0(x_0)} + \sum_{n=1}^{N-1} \underbrace{\log\left(P_{x_{n-1} x_n}\right)}_{-V_n(x_n, x_{n-1})}$

$p(x) = 1 \exp\left\{-\left(V_0(x_0) + \sum_{n=1}^{N-1} V_n(x_n, x_{n-1})\right)\right\}$

So the cliques are $\{n, n-1\}$ and $\{0\}$

$\Rightarrow \quad \partial n = \{n-1, n\} \cap \{0, \dots, N-1\}$

d) By HC Theorem

Gibbs Distribution $\Rightarrow$ $\left\{\begin{array}{l}\text{MRF} \\ \text{with } \partial n \\ \text{neighbourhood}\end{array}\right\}$

e) Since $\forall i,j \ P_{ij} > 0 \Rightarrow$ irreducible
aperiodic

$\left.\begin{array}{l}\text{irreducible} \\ \text{aperiodic} \\ \text{finite}\end{array}\right\} \Rightarrow$ ergodic

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stationary distribution given by

$$\underbrace{\pi = \pi P}_{\text{Full balance equations}}$$

f)

$$P\{X_{n-1}=i, X_n=j\} = \pi_i P_{ij}$$

$$P\{X_{n-1}=i \mid X_n=j\} = \frac{\pi_i P_{ij}}{\sum_k \pi_k P_{kj}} = Q_{ji}$$

$$Q_{ji} = \frac{\pi_i P_{ij}}{\pi_j}$$

g) Yes because π and P are not functions of time,

$\Rightarrow Q_{ji}$ is not a function of time.

h) reversible $\Leftrightarrow$ $P_{ij} = Q_{ij}$

$$P_{ij} = \frac{\pi_j P_{ji}}{\pi_i}$$

$$\pi_i P_{ij} = \pi_j P_{ji} \Leftarrow \text{detailed balance equations}$$

Problem 3.(36pt)

Let X, N, and Y be Gaussian random vectors such that $X \sim N(0, R_x)$ and $W \sim N(0, R_w)$, and let θ be a deterministic vector.

a) First assume that $Y = \theta + W$, and calculate the ML estimate of θ given Y .

For the next parts, assume that $Y = X + W$.

b) Calculate an expression for $p_{x|y}(x|y)$, the conditional density of X given Y.

c) Calculate the MMSE estimate of X when $Y = X + W$.

d) Calculate an expression for the conditional variance of X given Y.

a) MLE $\hat{\theta}_{ML} = Y$

b)

$$p(y|x) = \frac{1}{2} \exp\left\{\frac{-1}{2}(y-x)^t R_w^{-1} (y-x)\right\}$$

$$p(x) = \frac{1}{2} \exp\left\{-\frac{1}{2} x^t R_x^{-1} x\right\}$$

$$p(x|y) = \frac{1}{Z(y)} \exp\left\{-\frac{1}{2}\left((x-y)^t R_w^{-1} (x-y) + x^t R_x^{-1} x\right)\right\}$$

$$= \frac{1}{2} \exp\left\{ (x-\mu)^t R_{x|y}^{-1} (x-\mu)\right\}$$

$$R_{x|y}^{-1} = R_w^{-1} + R_x^{-1}$$

$$R_{x|y} = \left(R_w^{-1} + R_x^{-1}\right)^{-1}$$

$$(x-y)^t R_w^{-1} + x^t R_x^{-1} = (x-\mu)(R_w^{-1} + R_x^{-1})$$

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$$y^t R_w^{-1} = \mu(R_w^{-1} + R_x^{-1})$$

$$\mu = (R_w^{-1} + R_x^{-1})^{-1} R_w^{-1} y$$

$$= (I + R_w R_x^{-1})^{-1} y$$

c) $$\hat{\mu} = (R_w^{-1} + R_x^{-1})^{-1} R_w^{-1} y$$

$$= (I + R_w R_x^{-1})^{-1} y = R_x (R_w + R_x)^{-1} Y$$

d) $$R_{x|y} = (R_w^{-1} + R_x^{-1})^{-1} = R_w (R_w + R_x)^{-1} R_x$$

Problem 4.(10pt)

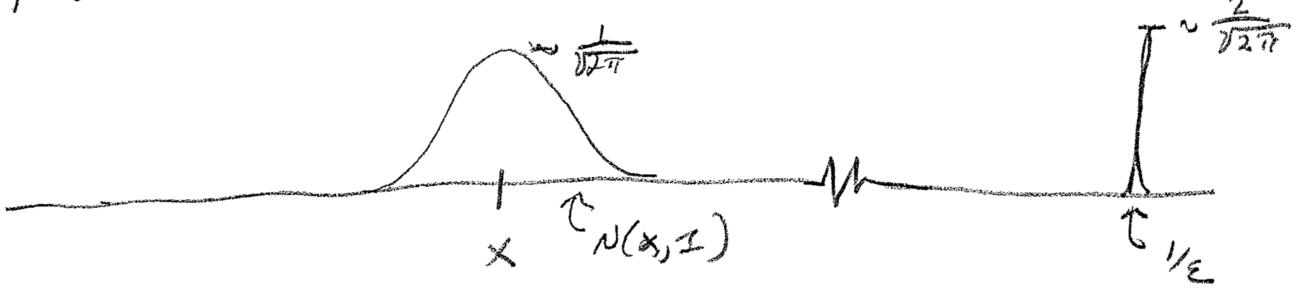
Let Y and X be random variables, and let Y_{MAP} and Y_{MMSE} be the MAP and MMSE estimates respectively of Y given X. Pick distributions for Y and X so that the MAP estimator is very "poor", but the MMSE estimator is "good".

$$Y = X + W$$

$$p(w) = (1-2\varepsilon)\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}w^2\right\} + 2\varepsilon\frac{1}{\sqrt{2\pi}\,\varepsilon}\exp\left\{-\frac{1}{2\varepsilon^2}\left(w-1/\varepsilon\right)^2\right\}$$

let $\varepsilon \to 0$

$p(y|x)$



$$\hat{Y}_{MAP} = 1/\varepsilon$$

$$\hat{Y}_{MMSE} \cong X$$