EE 641 Final Exam

Fall 2010

Name: ______________________________

Instructions

- This exam contains 4 problems worth a total of 154 points.
- You may not use any notes, textbooks, or calculators.
- Answer questions precisely and completely. Credit will be subtracted for vague answers.

Good luck.

Problem 1. (36pt)

Consider the p dimensional random vector, X, with distribution

$$p(x) = \frac{1}{z} \exp\left\{ -\frac{1}{2} x^t B x \right\}$$

a) Give an expression for the normalizing constant z.

b) Assuming that X is a Markov random field with neighborhood system ∂s for $s \in \{1, \cdots, p\}$, then specify which values of B must be zero.

c) Compute an expression for the conditional density $p(x_s | x_r \text{ for } r \neq s)$.

d) Compute an expression for the marginal density $p(x_s)$.

Problem 2.(72pt)

Let X_n for $n = 0$ to $N - 1$ be a discrete-valued Markov chain with homogeneous transition probability matrix P and initial probability density given by

$$\pi_i^{(0)} = P\{X_0 = i\} \ .$$

Furthermore, assume all the entries of P and $\pi^{(0)}$ are strictly positive and that the number of states of the Markov chain are finite.

a) Write the expression for the probability of the sequence $\{X_n\}_{n=0}^{N-1}$.

b) Write an algorithmic recursion for the probability

$$\pi_i^{(n)} = P\{X_n = i\} \ .$$

(Hint: The recursion should start at $n = 1$ and increment until $n = N - 1$.)

c) Show that that the discrete density, $p(x) = P\{X = x\}$, is a Gibbs distribution with neighborhood system given by

$$\partial n = \{n - 1, n + 1\} \cap \{0, \cdots, N - 1\} \ .$$

d) Use the Hammersley-Clifford Theorem to Prove that X is a Markov random field with neighborhood system ∂n.

e) Does a stationary distribution exist for this Markov chain? If it does exist, specify a set of equations (or a matrix equation) which can be used to determine the stationary distribution? What is the name of this set of equations?

For the following parts, let π be the stationary distribution for the Markov chain, and assume that $\pi^{(0)} = \pi$.

f) Derive an explicit and closed form expression for the transition probabilities of the time-reversed Markov chain.

$$Q_{j,i} = P\{X_n = i | X_{n+1} = j\} \ .$$

g) Is the time reversed Markov chain homogeneous? Why or why not?

h) Derive conditions for the Markov chain to be reversible in terms of π and P.

Problem 3.(36pt)

Let X, N, and Y be Gaussian random vectors such that $X \sim N(0, R_x)$ and $W \sim N(0, R_w)$, and let θ be a deterministic vector.

a) First assume that $Y = \theta + W$, and calculate the ML estimate of θ given Y .

For the next parts, assume that $Y = X + W$.

b) Calculate an expression for $p_{x|y}(x|y)$, the conditional density of X given Y.

c) Calculate the MMSE estimate of X when $Y = X + W$.

d) Calculate an expression for the conditional variance of X given Y.

Problem 4.(10pt)

Let Y and X be random variables, and let Y_{MAP} and Y_{MMSE} be the MAP and MMSE estimates respectively of Y given X. Pick distributions for Y and X so that the MAP estimator is very "poor", but the MMSE estimator is "good".