1)

$p(x|\theta) = \theta^k (1-\theta)^{N-k} \quad k = \#\text{ of } x_i = 1$

$\hat{\theta} = \arg\max_{\theta} \log p(x|\theta)$

$= \arg\max_{\theta} \; k \log\theta + (N-k)\log(1-\theta)$

$\frac{d}{d\theta} \log p(x|\theta) = 0 \Rightarrow \theta = \frac{k}{N}$

2)

a) $\hat{\theta} = Y$

b) $\log p(x|y) = \log p(y|x) + \log p(x) + c$

$$= -(y-x)^* R_n^{-1} (y-x) - x^* R_x^{-1} x + c$$

$$= -(x-\mu)^* R^{-1} (x-\mu) + c''$$

where $R^{-1} = (R_n^{-1} + R_x^{-1})$

$$\mu = (R_n^{-1} + R_x^{-1})^{-1} R_n^{-1} y$$

$$\hat{X}_{MMSE} = (R_n^{-1} + R_x^{-1})^{-1} R_n^{-1} y$$

$$= (R_n (R_n^{-1} + R_x^{-1}))^{-1} y$$

$$= ((R_x + R_n) R_x^{-1})^{-1}$$

$$= R_x (R_x + R_n)^{-1} y$$

From class

$$p(y) = \prod_{n=1}^{N} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left\{ -\frac{1}{2\sigma^2} \left(y_n - \sum_{i=n-p}^{n-1} h_{n-i}\, y_i \right)^2 \right\}$$

Define

$$z_n = \begin{bmatrix} y_{n-1} \\ \vdots \\ y_{n-p} \end{bmatrix} \qquad h = \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix}$$

$$\log p(y) = -\frac{1}{2\sigma^2} \sum_{n=1}^{N} \left(y_n - h^t z_n \right)^2 - \frac{N}{2} \log(2\pi\sigma^2)$$

$$= \frac{-N}{2\sigma^2} \left(\hat{\sigma}_y^2 - 2h^t b + h^t R h \right) - \frac{N}{2} \log(2\pi\sigma^2)$$

where $\hat{\sigma}_y^2 = \frac{1}{N} \sum_{n=1}^{N} y_n^2$

$$b = \frac{1}{N} \sum_{n=1}^{N} y_n z_n$$

$$R = \frac{1}{N} \sum_{n=1}^{N} z_n z_n^t$$

$$(\hat{\sigma}^2_{ML}, \hat{h}_{ML}) = \underset{(\sigma^2, h)}{\text{argmax}} \log p(y \mid \sigma^2, h)$$

$$= \underset{(\sigma^2, h)}{\text{argmax}} \left\{ -\frac{N}{2\sigma^2}\left(\hat{\sigma}_y - 2h^* b + h^* R h\right) - \frac{N}{2}\log(2\pi\sigma^2) \right\}$$

- Maximize with respect to h first.

$$\nabla_h \log p(y) = \frac{-1}{N}\nabla_h \left(h^* R h - 2 b^* h \right)$$

$$= \frac{-1}{N}\left(2 h^* R - 2 b^*\right) = 0$$

$$\hat{h}_{ML} = R^{-1} b$$

- Compute ML estimate of σ^2

$$\log p(y \mid \hat{h}_{ML}) = -\frac{1}{2\sigma^2} \sum_{n=2}^{N} (y_n - \hat{h}_{ML}^* z_n)^2 - \frac{N}{2}\log(2\pi\sigma^2)$$

$$= -\frac{N}{2\sigma^2} \hat{\sigma}_x^2 - \frac{N}{2}\log(2\pi\sigma^2)$$

where $\hat{\sigma}_x^2 \triangleq \frac{1}{N}\sum_{n=2}^{N} \underbrace{(y_k - \hat{h}^* z_k)^2}_{x_n^2}$

a) $1 - H(\omega) = 1 - \rho e^{-j\omega}$

$$S_x(\omega) = \frac{\sigma_c^2}{|1-H(\omega)|^2}$$

$$= \frac{\sigma_c^2}{1+\rho^2} \frac{1}{1 - 2\frac{\rho}{1+\rho^2}\cos(\omega)}$$

$$R_x(n) = Z^{-1}\left\{\frac{\sigma_c^2}{(1-\rho z^{-1})(1-\rho z)}\right\}$$

on unit circle $|\rho| < 1$

$$= \frac{\sigma_c^2}{1-\rho^2}\left\{\rho^n u(n) + \rho^{-n} u(-1-n)\right\}$$

$$p(y) = (2\pi)^{-N/2} |A|^{1/2} \exp\{-\tfrac{1}{2} y^t A y\}$$

find A so that

$$p(y_n \mid y_i\ i \neq n) \sim N\left(\sum_{i=1}^{N} y_i g_{n-i},\ \sigma_{NC}^2\right)$$

$$\frac{d}{dy_n} \log p(y_n \mid y_i\ i \neq n) = \frac{d}{dy_n} \log p(y)$$

$$= -\sum_{i=1}^{N} y_i A_{in} = -A_{nn}\left(y_n + \sum_{i \neq n} y_i \frac{A_{in}}{A_{nn}}\right)$$

$$\Leftrightarrow \log p(y_n \mid y_i\ i \neq n) = -\frac{A_{nn}}{2}\left(y_n + \sum_{i \neq n} y_i \frac{A_{in}}{A_{nn}}\right)^2 + C$$

$$\Leftrightarrow p(y_n \mid y_i\ i \neq n) = \frac{1}{\sqrt{2\pi}} \sqrt{A_{nn}} \exp\left\{-\frac{A_{nn}}{2}\left(y_n + \sum_{i \neq n} y_i \frac{A_{in}}{A_{nn}}\right)^2\right\}$$

$$\Leftrightarrow \forall y_i \quad A_{nn} = \frac{1}{\sigma_{NC}^2}\ ;\ \sum_{i \neq n} y_i \frac{A_{in}}{A_{nn}} = -\sum_{i=1}^{N} y_i g_{n-i}$$

$$\Leftrightarrow \quad A = \frac{1}{\sigma_{NC}^2} B$$