EE 641 Final Exam

Fall 2004

Name: Key

Starting time: ______________

Ending time: ______________

Instructions

The following are important rules for this take home exam.

- Accurately fill in a start time and ending time for your exam.
- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period.
- Your are allowed to use all class notes and handouts, any material posted on the official course web page, and the course text book from EE600 or an equivalent graduate course. Your are not allowed to use supplementary information from the library, or publications not handed out in class or contained on the course web page.
- You must hand in your completed exam by Monday December 20 at 9:00AM.
- If you have any questions, call me at home 463-4378 (8:00AM to 8:00PM); or office 494-0340, or send email at bouman@purdue.edu.

Good luck.

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Problem 1.(25pt)

Let X be a Gaussian random vector with distribution $N(0, R_x)$, let W be an independent Gaussian random vector with distribution $N(0, R_w)$, and let $Y = X + W$.

20 a) Derive the conditional distribution $p(x|y)$.

7 b) What estimator, $\hat{X}$, minimizes the $E\left[||X - \hat{X}||^2\right]$?

6 c) What is the MAP estimate of X give Y?

a) $$p(y|x) = \frac{1}{(2\pi)^{N/2}} |R_w|^{1/2} \exp\left\{-\frac{1}{2}(y-x)^t R_w^{-1}(y-x)\right\}$$

$$p(x) = \frac{1}{(2\pi)^{N/2}} |R_x|^{1/2} \exp\left\{-\frac{1}{2} x^t R_x^{-1} x\right\}$$

$$\log p(x,y) = -\frac{1}{2}(y-x)^t R_w^{-1}(y-x) - \frac{1}{2} x^t R_x^{-1} x + \text{const}$$

$$\log p(x|y) = -\frac{1}{2}(x-\mu_x)^t B (x-\mu_x) + \text{const}$$

Note that $$\nabla_x \nabla_x \log p(x,y) = \nabla_x \nabla_x \log p(x|y)$$

$$\Rightarrow \quad B = R_w^{-1} + R_x^{-1}$$

$$\nabla_x \log p(x,y) = \nabla_x \log p(x|y)$$

$$\Rightarrow \quad (y-x)^t R_w^{-1} - x^t R_x^{-1} = -(x-\mu_x)^t B$$

$$x^t(R_w^{-1} + R_x^{-1}) - y^t R_w^{-1} = x^t B - \mu_x^t B$$

if $x = 0$

$$\Rightarrow \quad -y^t R_w^{-1} = -\mu_x^t B \qquad (B = B^t \quad R_w = R_w^{-1})$$

$$\mu_x = B^{-1} R_w^{-1} Y$$

$$\mu_x = (R_w^{-1} + R_x^{-1})^{-1} R_w^{-1} Y$$

$$= \left[R_w^{-1}(I + R_w R_x^{-1})\right]^{-1} R_w^{-1} Y$$

$$= (I + R_w R_x^{-1})^{-1} R_w R_w^{-1} Y$$

$$= \left[(R_x + R_w) R_x^{-1}\right]^{-1} Y$$

$$\boxed{\mu_x = R_x (R_x + R_w)^{-1} Y}$$

$$R_{x|y} = B^{-1}$$

$$= (R_x^{-1} + R_w^{-1})^{-1}$$

$$= \left[R_x^{-1}(I + R_x R_w^{-1})\right]^{-1}$$

$$= (I + R_x R_w^{-1})^{-1} R_x$$

$$= \left[(R_w + R_x) R_w^{-1}\right]^{-1} R_x$$

$$\boxed{R_{x|y} = R_w (R_x + R_w)^{-1} R_x}$$

$$p(x|y) \sim N(\mu_x, R_{x|y})$$

b) $\hat{X} = R_x (R_x + R_w)^{-1} Y$

c) $\hat{X} = R_x (R_x + R_w)^{-1} Y$

Problem 2.(~~25~~ 33pt) Let X and Y be random variables, and let the conditional density of X given Y be

$$p(x|y) = (1-\epsilon)\frac{1}{\sqrt{2\pi}}\exp\left\{\frac{1}{2}x^2\right\} + \epsilon\frac{1}{\sqrt{2\pi\epsilon^2}}\exp\left\{\frac{1}{2\epsilon^4}\left(x - \frac{1}{\sqrt{\epsilon}}\right)^2\right\}$$

where ϵ is a small value greater than 0.

4 a) Sketch the distribution $p(x|y)$.

4 b) What is the name of this distribution that was used in lectures and labs?

4 c) Varify that $\int_{-\infty}^{\infty} p(x)dx = 0$

4 d) Compute $\bar{X}_\epsilon$, the MMSE estimator of X given Y as a function of ϵ.

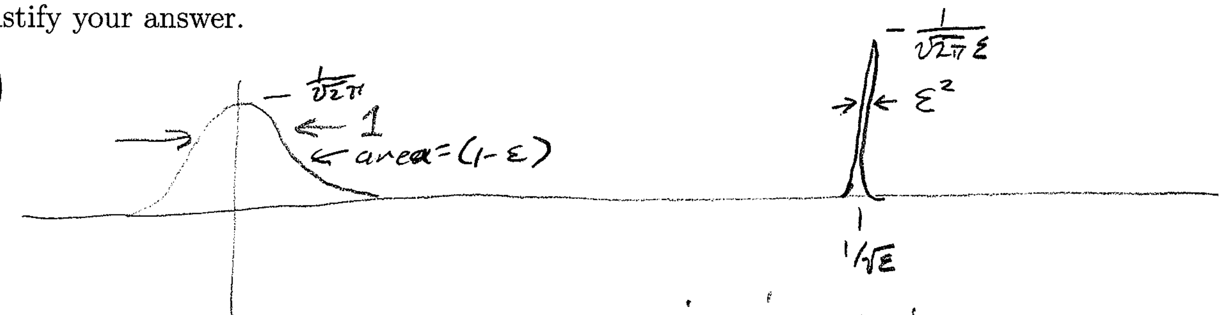
4 e) Compute $\lim_{\epsilon\to 0}\bar{X}_\epsilon$.

4 f) Approximately compute $\hat{X}_\epsilon$, the MAP estimate of X given Y as ϵ becomes small.

4 g) Compute $\lim_{\epsilon\to 0}\hat{X}_\epsilon$.

5 h) As ϵ becomes very small, which estimator is better for this example, MMSE or MAP? Justify your answer.

a)



b) Gaussian mixture distribution

c) $\int_{-\infty}^{\infty} p(x)\,dx = (1-\varepsilon) + \varepsilon = 1$

d) $E[X|Y] = (1-\varepsilon)0 + \varepsilon\frac{1}{\sqrt{\varepsilon}}$

$= \sqrt{\varepsilon}$

e) $\lim_{\varepsilon\to 0}\bar{X} = \lim_{\varepsilon\to 0}\sqrt{\varepsilon} = 0$

f) $\hat{X}_\varepsilon \simeq \frac{1}{\sqrt{\varepsilon}}$

g) $\lim_{\varepsilon\to 0}\hat{X}_\varepsilon = \infty$

h) As $\varepsilon \to 0$ the probability that X is in the second "bump" $\to 0$. So MMSE is better than MAP.

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Problem 3.(~~34~~ 34pt)

Let X and Y be random variables with joint density $p(x, y|\theta) > 0$ for all $(x, y) \in \mathbb{R}^2$, and let $p_y(y|\theta)$ be the marginal density of Y which is assumed to be a continuously differentible function of θ. Furthermore, define

$$Q(\theta', \theta) = E\left[\log p(X, Y|\theta')|Y = y, \theta\right]$$

Prove that if $Q(\theta', \theta) > Q(\theta, \theta)$ then $p_y(y|\theta') > p_y(y|\theta)$

See class notes