EE 641 Midterm Exam

Fall 2004

Name: ________________________________ Starting time: ________________________

Ending time: ________________________

Instructions

The following is a take home exam.

- This exam contains 4 problems and should be completed in 75 minutes.
- Answer questions precisely and completely. Credit will be subtracted off for vague answers.
- You may not use any notes, textbooks, or calculators; but you may use a single sided single page of hand copied notes.

Good luck.

Problem 1. (20pt)

Let $\{X_n\}_{n=0}^N$ be a Markov Chain with

$$P\{X_0 = k\} = \pi_k$$

for $1 \leq k \leq M$, and

$$P\{X_n = m | X_{n-1} = l\} = P_{l,m}$$

for $1 \leq l, m \leq M$. Let π denote the $1 \times M$ vector of initial probabilities, and let P denote the $M \times M$ transition matrix.

a) Derive an expression, $p(x|\theta)$, the likelihood for the entire sequence.

b) Derive the ML estimate of $\theta = (\pi, P_{l,m})$ given $\{X_n\}_{n=0}^N$.

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Problem 2. (25pt)

Let $\{X_n\}_{n=0}^{N-1}$ be i.i.d. random variables with

$$P\{X_n = k\} = \pi_k$$

for $0 \leq k \leq M-1$.

Let $\{Y_n\}_{n=0}^{N-1}$ be random variables with conditional distribution

$$p(y|x,\theta) = \prod_{n=0}^{N-1} \frac{1}{\mu_{x_n}} \exp\left\{-y_n/\mu_{x_n}\right\} \mathbf{1}(y_n \geq 0)$$

where $\mathbf{1}(y_n \geq 0)$ is function which is 1 when $y_n \geq 0$ and zero otherwise, and $\theta = (\pi, \mu)$

a) Derive $p(y, x|\theta)$.

b) Derive $f(x_n|y_n, \theta) = p(x_n|y_n, \theta)$.

c) Derive the ML estimate of θ given both Y and X.

d) Derive the EM update equations for estimation of θ from Y.

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Problem 3. (30pt)

Let X be an N pixel Gaussian MRF with density

$$p(x) = \frac{1}{(2\pi\sigma^2)^{N/2}} |B|^{N/2} \exp\left\{ -\frac{1}{2\sigma^2} x^t B x \right\}$$

Furthermore, let Y be an N pixel random field with pixels that are conditionally i.i.d. and Poisson given X, and

$$E[Y|X] = X$$

a) Calculate the maximum likelihood estimate of σ^2 given X and Y.

b) Calculate $p(y, x|\sigma)$, the joint probability density of Y and X given σ.

c) Calculate an expression for the MAP estimate of X given Y.

d) Derive an expression for the gradient descent update of the MAP cost functional.

e) Derive an expression for the ICD update of a signal pixel s.

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Problem 4. (25pt)

Consider the cost functional

$$||y - Hx||^2 + x^t Bx$$

which takes on its unique minimum value at

$$\hat{x} = \arg \min_{x \in \mathbb{R}^N} \left\{ ||y - Hx||^2 + x^t Bx \right\}$$

a) Derive the gradient descent algorithm for this minimization problem, and show it has the form of the difference equation

$$x^{(k+1)} = (I - \omega P)\, x^{(k)} + \omega H^t y$$

where $P = H^t H + B$.

Further assume that $P = T^t \Lambda T$ were T is an orthonormal matrix, and Λ is a diagonal matrix with positive entrees $\lambda_1, \cdots, \lambda_N$.

b) Let $e^{(k)} = x^{(k)} - \hat{x}$, and derive a difference equation for $e^{(k)}$.

c) Determine the range of values of ω that will allow stable convergence of the gradient descent algorithm.

d) Explain what the trade-offs are in the selection of ω.

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