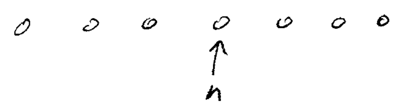
Markov Random Fields

Objective - Use the noncausal conditional expectation to model a signal / image.



$$p_n(x_n \mid x_i \;\; i \neq n) = p_n(x_n \mid x_{\partial n})$$

$x_{\partial n}$ - neighbors of x_n

How do we write down the distribution for x?

$$p(x) \neq \prod_{n=1}^{N} p_n(x_n \mid x_{\partial n}) \quad ? \quad NO!$$

Neighborhoods

Define

$$\partial s = \{ r : r \text{ is a neighbor of } s \}$$

Example

4pt neighborhood $\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \\ & \circ & \end{array}$

8pt neighborhood $\begin{array}{ccc} \circ & \circ & \circ \\ \circ & \times & \circ \\ \circ & \circ & \circ \end{array}$

- Every neighborhood must have the following properties

$$\left.\begin{array}{l} 1)\quad s \notin \partial s \\ \\ 2)\quad s \in \partial r \Longleftrightarrow r \in \partial s \end{array}\right\} \text{important}$$

Notation:

$$X_{\partial s} \triangleq \{ X_i : i \in \partial s \}$$

Example Is $\begin{array}{cc} \bullet & \bullet \\ \bullet & \times \end{array}$ a neighborhood system?

NO! $\begin{array}{cc} a & b \\ c & d \end{array}$ $\underbrace{a \in \partial d \text{ but } d \notin \partial a}_{\text{not allowed}}$

$\Rightarrow$ must be noncausal

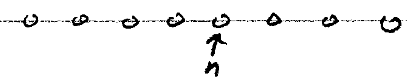
Lecture 15

- Markov Random Field (MRF)

Defn: X is an MRF with neighborhood system ∂s if

$$p(x_s \mid x_i \; i \neq s) = p(x_s \mid x_{\partial s})$$

Example 1-D



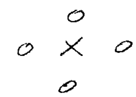
$$p(x_n \mid x_i \; i \neq n) = p(x_n \mid x_{n-1}, x_{n+1})$$

$$\partial n = \{n-1, n+1\}$$

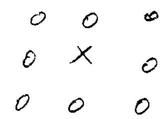
Gibbs Distribution

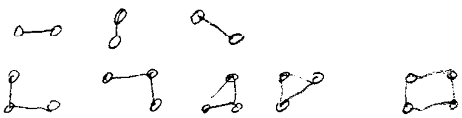
Defn $c \subset S$ is a clique on the lattice S with neighborhood system ∂S if $\forall a, b \in c$ $a \in \partial b$

Example

1) 4pt neighbors 

   cliques 

2) 8pt neighbors 

   cliques 

$$\mathcal{C} \triangleq \{ c : c \text{ is a clique} \}$$

Defn $p(x)$ is a Gibbs distribution if it can be written in the form

$$p(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{T} \sum_{c \in \mathcal{C}} V_c(x_c) \right\}$$

Defn X is strictly positive if

for all x $p(x) > 0$.

The Hammersley Clifford theorem:

Let X be a strictly positive random field.
Then X is an MRF $\Leftrightarrow$ $p(x)$ is a Gibbs distribution

proof

$(\Leftarrow)$

$$p(x_s \mid x_i\ i \neq s) = \frac{p(x)}{p(x_i\ i \neq s)}$$

$$= \frac{\frac{1}{z} \exp\left\{ - \sum_{c \in C} V_c(x_c) \right\}}{\sum_{x_s} \frac{1}{z} \exp\left\{ - \sum_{c \in C} V_c(x_c) \right\}}$$

Define $C_s = \{ c : c \in C \text{ and } s \in c \}$

$\uparrow$ cliques with the point s

$\bar{C}_s = C - C_s$

$\uparrow$ cliques without the point s

$p(x_s \mid x_i \; i \neq s) =$

$$\frac{\cancel{\frac{1}{2}\exp\{-\sum_{c\in \bar{C}_s} V_c(x_c)\}}\exp\{-\sum_{c\in C_s} V_c(x_c)\}\cancel{\exp\{-\sum_{c\in \bar{C}_s} V_c(x_c)\}}}{\cancel{\frac{1}{2}\exp\{-\sum_{c\in \bar{C}_s} V_c(x_c)\}}\sum_{x_s}\exp\{-\sum_{c\in C_s} V_c(x_c)\}}$$

$$= \frac{\exp\{-\sum_{c\in C_s} V_c(x_c)\}}{\sum_{x_s}\exp\{-\sum_{c\in C_s} V_c(x_c)\}}$$

$$= f(x_s, x_{\partial s})$$

$$= p(x_s \mid x_{\partial s})$$

$\boxed{\text{Lecture 16}}$

$(\Rightarrow)$

1) Define
$$Q(x) = \log\left(\frac{p(x)}{p(0)}\right)$$

Then there exist functions G so that

$$
\begin{aligned}
Q(x) = & \sum_{1\le i\le n} x_i G_i(x_i) \\
& + \sum\sum_{1\le i<j\le n} x_i x_j G_{ij}(x_i, x_j) \\
& + \sum\sum\sum_{1\le i<j<k\le n} x_i x_j x_k G_{ijk}(x_i, x_j, x_k) \\
& \quad \vdots \\
& + x_1 x_2 \dots x_n G_{123\dots n}(x_1, x_2, \dots x_n)
\end{aligned}
$$

2) Define $\vec{X}_n = \{x_1, \dots x_{n-1}, 0, x_{n+1}, \dots x_n\}$

then $Q(x) - Q(\vec{X}_n) = \log \dfrac{p(x_n, x_i\ i\neq n)}{p(X_n=0, x_i\ i\neq n)}$

$$= \log \frac{p(x_n \mid X_{\partial n})\, p(x_i\ i\neq n)}{p(0\mid X_{\partial n})\, p(x_i\ i\neq n)}$$

$$= \log\left(\frac{p(x_n\mid X_{\partial n})}{p(0\mid X_{\partial n})}\right) = f(x_n, x_{\partial n})$$

2) If we subtract expressions for $Q(x)$ and $Q(\vec{X}_1)$, then terms not containing $\vec{X}_1$ must cancell

$$Q(x) - Q(\vec{X}_1) =$$

$$= X_1 \Big\{ G_1(X_1) + \sum_{2 \le j \le n} x_j \, G_{1j}(X_1, X_j)$$

$$+ \ldots + X_2 X_3 \ldots x_n \, G_{12\ldots n}(X_1, \ldots X_n) \Big\}$$

3. Choose $l \notin \partial 1$

Let $x_i = 0$ for $i \neq 1, l$

$$Q(x) - Q(x_1) = x_1 \{ G_1(x_1) + x_l G_{1l}(x_1, x_l) \}$$
$$= f(x_1, x_{\partial 1})$$

$\Rightarrow G_{1l} = 0$

4. Choose $l \in \partial 1$, $m \notin \partial 1$

Let $x_i = 0$ for $i \neq l, m, 1$

$$Q(x) - Q(x_1) = x_1 \{ G_1(x_1) + x_l G_{1l}(x_l) + x_l x_m G_{1lm}(x_1, x_l, x_m) \}$$

$$= f(x_1, x_{\partial 1})$$

So $x_1 x_l x_m G_{1lm}(x_1, x_l, x_m) = f(x_1, x_{\partial 1})$
$= f(x_1, x_l)$

$\Rightarrow \; = x_1 x_l f'(x_1, x_l)$ from some $f'(x_1, x_l)$

but since when $x_m = 0 \Rightarrow x_1 x_l x_m G_{1lm} = 0$

$$\Rightarrow x_1 x_l f'(x_1, x_l) = 0$$

5. By induction,

If $\kappa \notin \partial 1$ then

$$G_{1, i_1, \ldots, \kappa, \ldots i_n} = 0$$

6. By a similar argument

if $i_\kappa \notin \partial i_j$ then

$$G_{i_1 \ldots i_n} = 0$$

7)

$$V_c(x_c) \triangleq -G_c(x_c) \prod_{s \in c} x_s$$

$$Q(x) = \sum_{c \in C} -V_c(x_c)$$

$$p(x) = p(\theta) \exp\left\{-\sum_{c \in C} V_c(x_c)\right\}$$

$$= \frac{1}{Z} \exp\left\{\frac{-1}{T} \sum_{c \in C} V_c(x_c)\right\}$$

Q.E.D.

Example GMRF

$$p(x) = \frac{1}{\sqrt{2\pi}^N} |B|^{-1/2} \exp\{-\tfrac{1}{2} X^T B X\}$$

$$-\frac{1}{2} X^T B X = \frac{1}{2} \sum_{i,j} X_i B_{ij} X_j$$

$$= \sum_{\{i,j\} \in C} X_i B_{ij} X_j + \sum_{i \in S} X_i^2 B_i$$

$$p(x) = \frac{1}{Z} \exp\Big\{ \sum_{\{i,j\} \in C} X_i B_{ij} X_j + \sum_{i \in S} B_i X_i^2 \Big\}$$

Only requires clique pairs
and singles

Lecture 17

Example

$\{X_i\}_{i=1}^{N}$ Markov Chain with strictly positive distribution.

$$p(X) = \prod_{i=2}^{N} p_i(x_i \mid x_{i-1})\, p_1(x_1)$$

Define $V_i(x_i, x_{i-1}) = -\log p(x_i \mid x_{i-1})$

$$V_1(x_1) = -\log p(x_1)$$

$$p(x) = \exp\left\{-\sum_{i=2}^{N} V_i(x_i, x_{i-1}) - V_1(x_1)\right\}$$

Gibbs distribution $\Rightarrow$ MRF

If $p_i(x_i \mid x_{i-1}) = \frac{\rho}{M-1} t(x_i, x_{i-1}) + (1-\rho)\,\delta(x_i, x_{i-1})$

$x_i \in [1, M]$

$$-\log p(x_i \mid x_{i-1}) = -\log\left\{\frac{\rho}{1-\rho}\frac{1}{M-1} t(x_i, x_{i-1}) + \delta(x_i, x_{i-1})\right\} - \log(1-\rho)$$

$$= \log\left((M-1)\frac{(1-\rho)}{\rho}\right) t(x_i, x_{i-1}) - \log(1-\rho)$$

$$= V_i(x_i, x_{i-1})$$

$$\beta \triangleq \log\left((M-1)\frac{(1-\rho)}{\rho}\right)$$

$$p(x) = \frac{1}{Z}\exp\left\{-\sum_{i=2}^{N} \beta\, t(x_i, x_{i-1}) + V_1(x_1)\right\}$$

$$p(x_n \mid x_i \; i \neq n) = \frac{p(x)}{\sum_{x_n} p(x_n, x_i \; i \neq n)}$$

$$= \frac{\frac{1}{Z} \exp\left\{ - \sum_{i=2}^{N} \beta \, t(x_i, x_{i-1}) + V_1(x_1) \right\}}{\sum_{x_n} \frac{1}{Z} \exp\left\{ - \sum_{i=2}^{N} \beta \, t(x_i, x_{i-1}) + V_1(x_1) \right\}}$$

$$= \frac{\exp\left\{ -\beta \left(t(x_n, x_{n-1}) + t(x_{n+1}, x_n) \right) \right\}}{\sum_{k=0}^{M-1} \exp\left\{ -\beta \left(t(k, x_{n-1}) + t(x_{n+1}, k) \right) \right\}}$$