The Ising Model

- Motivated by model for Magnetism

$$X_s = \begin{cases} 1 & \text{North pole} \\ -1 & \text{South pole} \end{cases}$$

- MRF with 4 pt neighborhood

$$\begin{array}{ccc} & \circ & \\ \circ & \times & \circ \\ & \circ & \end{array}$$

- Let $r \in \partial s$ then

if $X_r X_s = 1 \Rightarrow$ poles are aligned $\Rightarrow$ low energy
$X_r X_s = -1 \Rightarrow$ pole are opposit $\Rightarrow$ high energy

Total energy $= U(X) = -\frac{J}{2} \sum_{\{r,s\} \in C} X_r X_s$

C - set of all cliques
J - physical constant

$$p(x) = P\{X = x\}$$

$$\text{Entropy} = S(p) = \sum_x -\log p(x)\, p(x)$$
$$= E[-\log p(X)]$$

$$\bar{\mathcal{E}} = \sum_x U(x)\, p(x) = E_p[U(X)]$$

If the system is in thermodynamic equilibrium then

$$p_e(x) = \underset{\substack{p \text{ s.t.} \\ E_p[U(X)] = \text{constant}}}{\operatorname{argmax}} S(p)$$

Maximize entropy while constraining energy $\Rightarrow$ Lagrange Multipliers

Solution

$$p(x) = \frac{1}{Z(T)} e^{-\frac{1}{KT} U(x)}$$

K - Boltzmann's constant
T - Temperature
$Z(T)$ - Partition function $= \sum_x e^{-\frac{1}{KT} U(x)}$

$$p(x) = \frac{1}{Z} \exp\left\{ -\frac{1}{KT} \left(-\frac{J}{2} \sum_{\{r,s\} \in C} x_r x_s \right) \right\}$$

$$= \frac{1}{Z} \exp\left\{ \frac{J}{2KT} \sum_{\{r,s\} \in C} x_r x_s \right\}$$

$$= \frac{1}{Z} \exp\left\{ \frac{J}{KT} \sum_{\{r,s\} \in C} \left(\frac{1}{2} - t(x_r, x_s) \right) \right\}$$

$$t(k, l) = \begin{cases} 1 & k \neq l \\ 0 & \text{o.w} \end{cases}$$

$$= \frac{1}{Z'} \exp\left\{ -\beta \sum_{\{r,s\} \in C} t(x_r, x_s) \right\}$$

$$\beta = \frac{J}{KT}$$

Note: Looks like the form of the 1-D Markov Chain

• What is the NonCausal dependence?

$$C_s = \{ c \in \mathcal{C} : x_s \in c \}$$
$$\bar{C}_s = \mathcal{C} - C_s$$

$$p(x_s \mid x_i \; i \neq s) = \frac{p(x_s, x_i \; i \neq s)}{\sum_{x_s} p(x_s, x_i \; i \neq s)}$$

$$= \frac{\frac{1}{z} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\} \exp\left\{ -\beta \sum_{\{i,j\} \in \bar{C}_s} t(x_i, x_j) \right\}}{\frac{1}{z} \exp\left\{ -\beta \sum_{\{i,j\} \in \bar{C}_s} t(x_i, x_j) \right\} \sum_{x_s} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}$$

$$= \frac{\exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}{\sum_{x_s} \exp\left\{ -\beta \sum_{\{i,j\} \in C_s} t(x_i, x_j) \right\}}$$

Define

$$V(x_s, X_{\partial s}) = \sum_{\{i,j\} \in C_s} t(x_i, x_j)$$

= Number of neighbors in $X_{\partial s}$ <u>not</u> equal to x_s

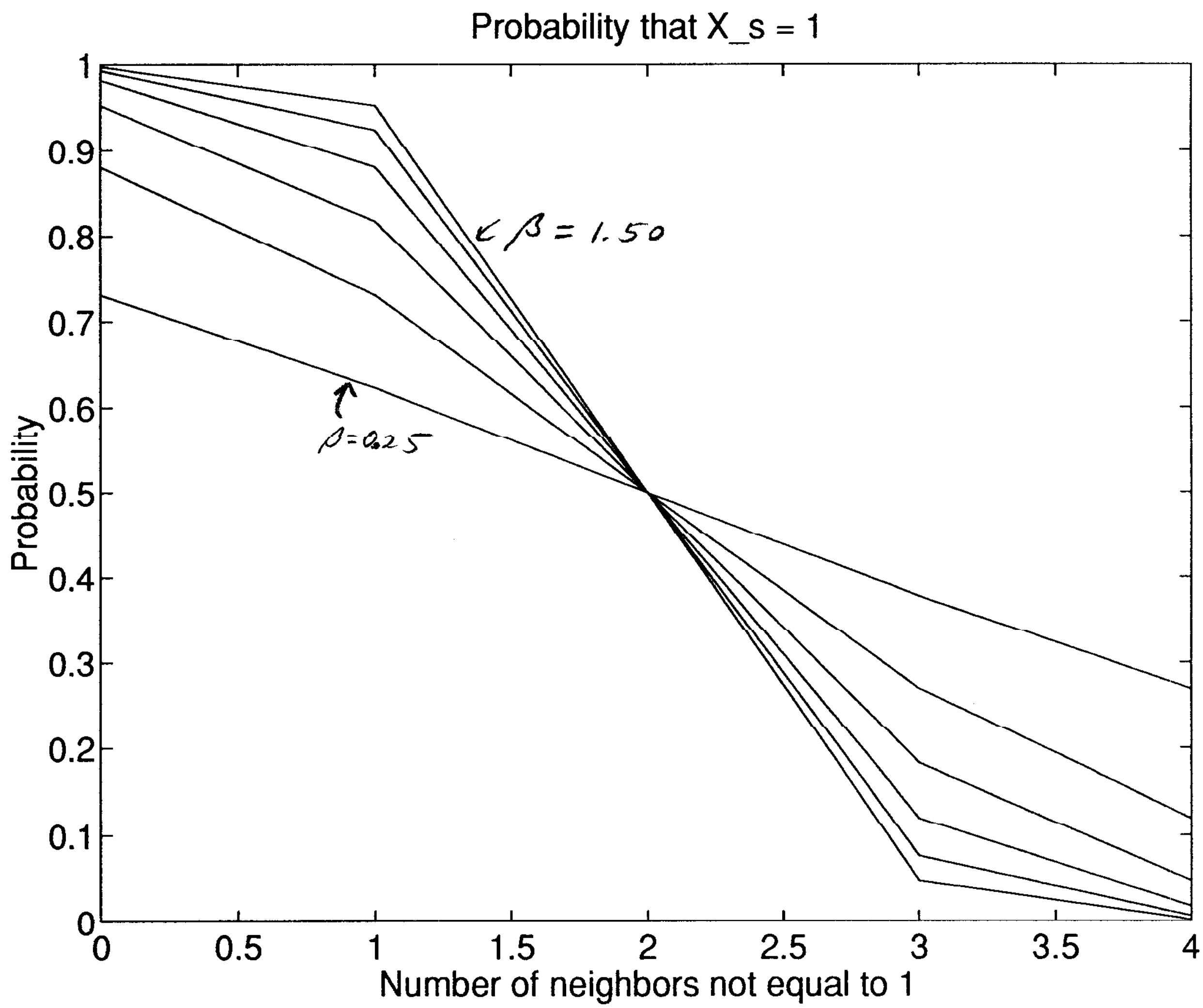
$p(x_s | x_i \; i \neq s) =$

$$\frac{\exp\{-\beta V(x_s, x_{\partial s})\}}{\exp\{-\beta V(x_s, x_{\partial s})\} + \exp\{-\beta V(-x_s, x_{\partial s})\}}$$

Notice that

$$V(-x_s, X_{\partial s}) = 4 - V(x_s, X_{\partial s})$$

$$p(x_s | x_i \; i \neq s) = \frac{1}{1 + \exp\{2\beta(V(x_s, x_{\partial s}) - 2)\}}$$



Probability that X_s = 1
Probability
Number of neighbors not equal to 1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0.5
1.5
2
2.5
3
3.5
4
β = 1.50
β = 0.25