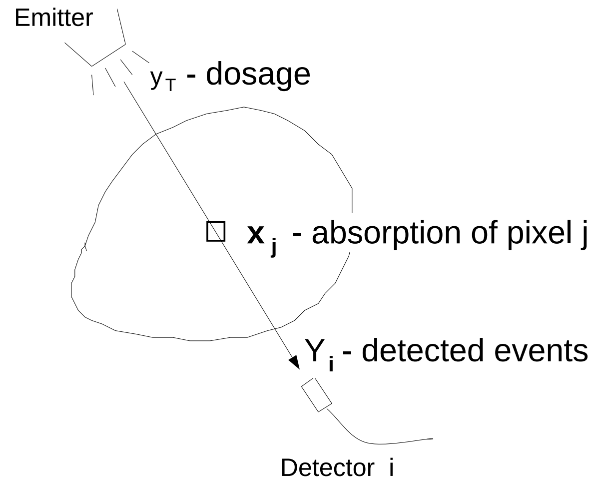
Application of Inverse Methods to Tomography

- Topics to be covered:
  - Tomographic system and data models
  - MAP Optimization
  - Parameter estimation

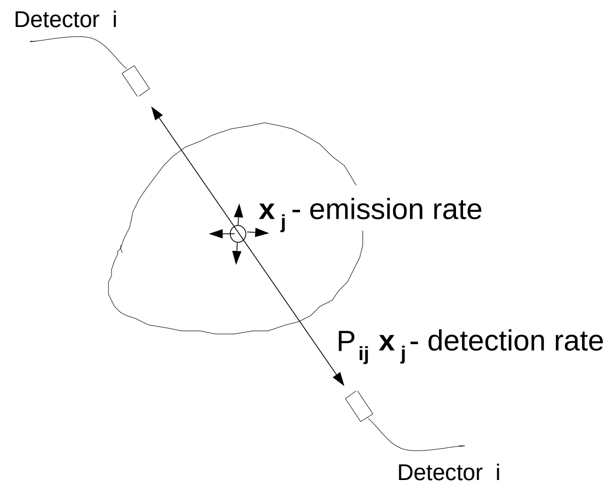
The Tomography Problem

- Recover image cross-section from integral projections

- Transmission problem



- Emission problem



Statistical Data Model[3]

- Notation
  - y - vector of photon counts
  - x - vector of image pixels
  - P - projection matrix
  - $P_{j,*}$ - j^{th} row of projection matrix
- Emission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-P_{i*}x \ + y_i \log\{P_{i*}x\} - \log(y_i!)\right)$$

- Transmission formulation

$$\log p(y|x) = \sum_{i=1}^{M} \left(-y_T e^{-P_{i*}x} + y_i(\log y_T - P_{i*}x) - \log(y_i!)\right)$$

- Common form

$$\boxed{\log p(y|x) = -\sum_{i=1}^{M} f_i(P_{i*}x)}$$

  - $f_i(\cdot)$ is a convex function
  - Not a hard problem!

Maximum A Posteriori Estimation (MAP)

- MAP estimate incorporates prior knowledge about image

$$\begin{aligned}\hat{x} &= \arg\max_{x} p(x|y) \\ &= \arg\max_{x>0} \left\{ -\sum_{i=1}^{M} f_i(P_{i*}x) - \sum_{k<j} b_{k,j}\, \rho(x_k - x_j) \right\}\end{aligned}$$

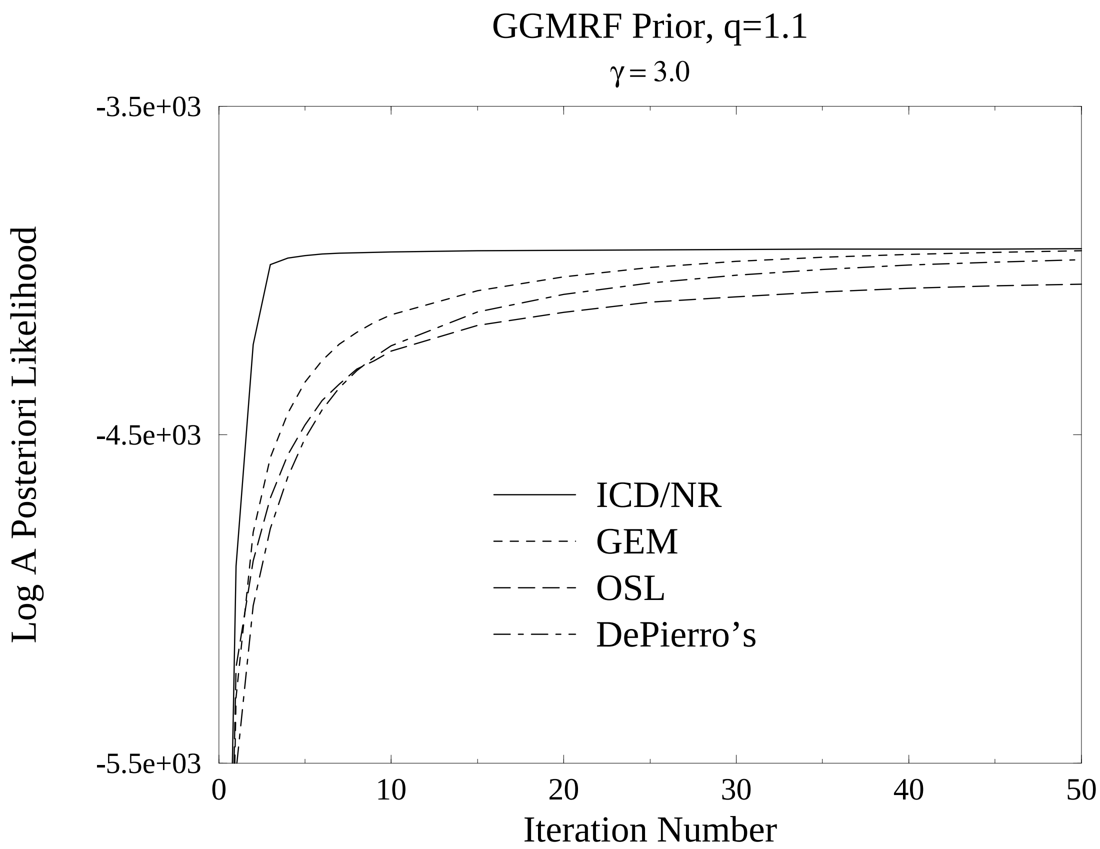
- Can be solved using direct optimization
- Incorporates positivity constraint

MAP Optimization Strategies

- Expectation maximization (EM) based optimization strategies
  - ML reconstruction[12, 10]
  - MAP reconstruction[8, 7, 9]
  - Slow convergence; Similar to gradient search.
  - Accelerated EM approach[6]
- Direct optimization
  - Preconditioned gradient descent with soft positivity constraint[5]
  - ICM iterations (also known as ICD and Gauss-Seidel)[3]

Convergence of ICM Iterations: MAP with Generalized Gaussian Prior $q = 1.1$

- ICM also known as iterative coordinate descent (ICD) and Gauss-Seidel



- Convergence of MAP estimates using ICD/Newton-Raphson updates, Green's (OSL), and Hebert/Leahy's GEM, and De Pierro's method, and a generalized Gaussian prior model with $q = 1.1$ and $\gamma = 3.0$.

Estimation of σ from Tomographic Data

- Assume a GGMRF prior distribution of the form

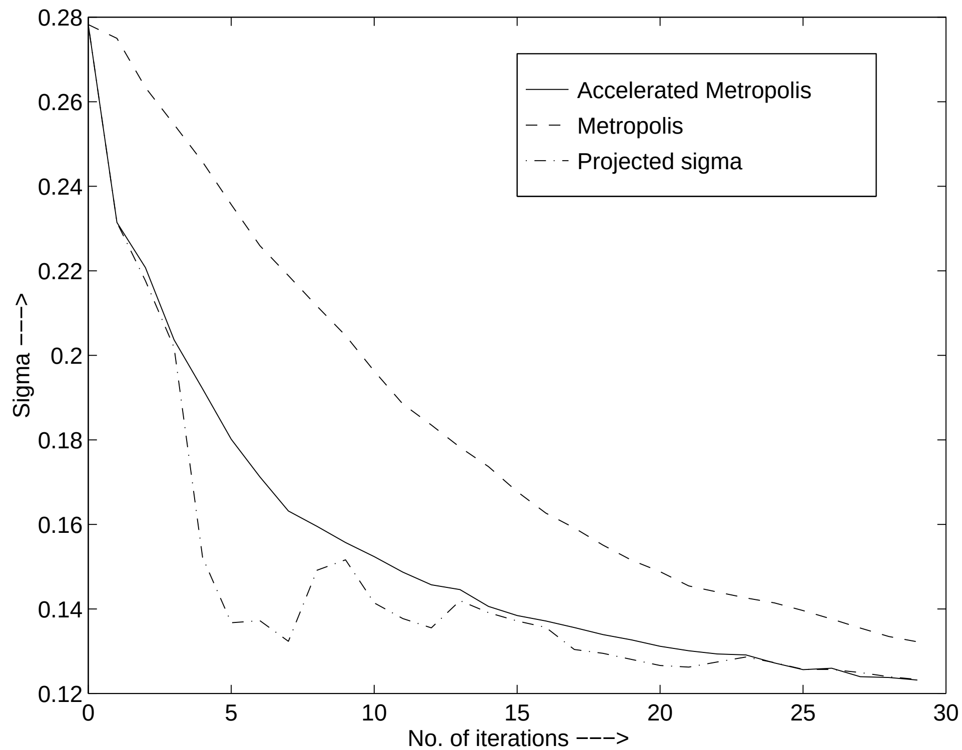
$$p(x) = \frac{1}{\sigma^N Z(1)} \exp\left\{\frac{1}{p\sigma^p} U(x)\right\}$$

- Problem: We don't know X!
- EM formulation for incomplete data problem

$$\begin{aligned} \sigma^{(k+1)} &= \arg\max_{\sigma} E\left\{\log p(X|\sigma)|Y = y, \sigma^{(k)}\right\} \\ &= \left(E\left\{\frac{1}{N} U(X)|Y = y, \sigma^{(k)}\right\}\right)^{1/p} \end{aligned}$$

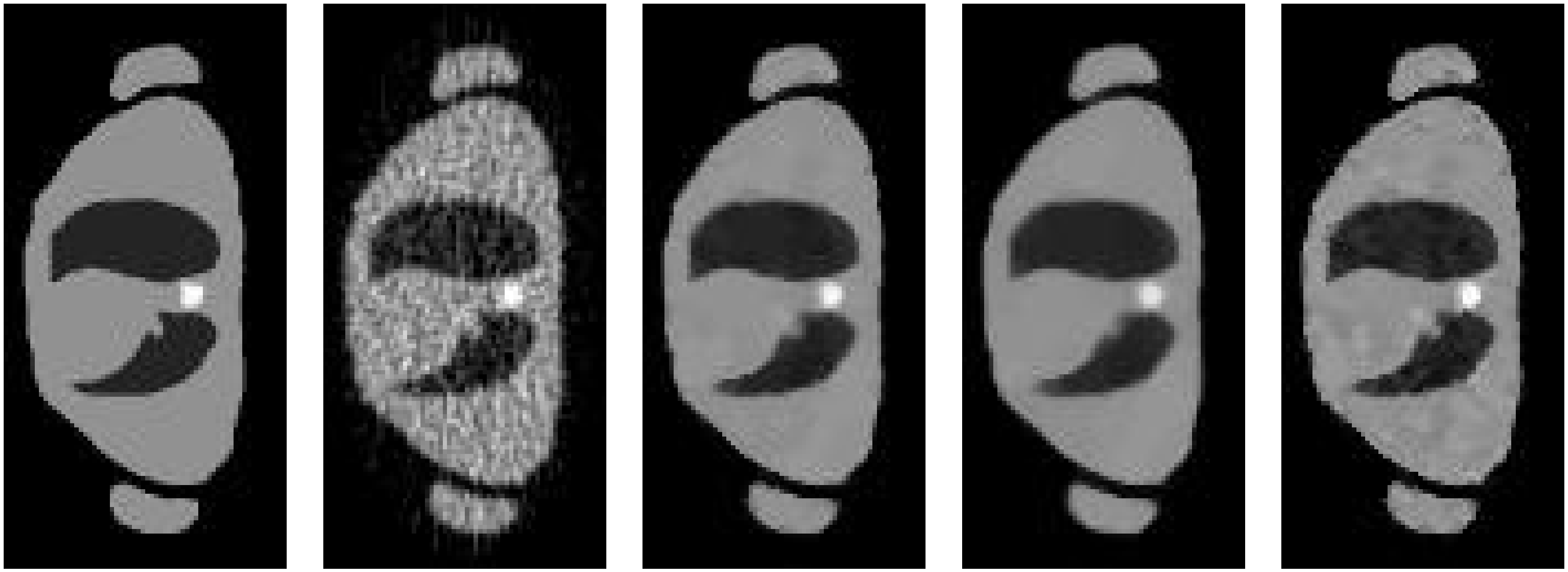
- Iterations converge toward the ML estimate.
- Expectations may be computed using stochastic simulation.

Example of Estimation of σ from Tomographic Data



- The above plot shows the EM updates for σ for the emission phantom modeled by a GGMRF prior ($p = 1.1$) using conventional Metropolis (CM) method, accelerated Metropolis (AM) and the extrapolation method. The parameter s denotes the standard deviation of the symmetric transition distribution for the CM method.

Example of Tomographic Reconstructions

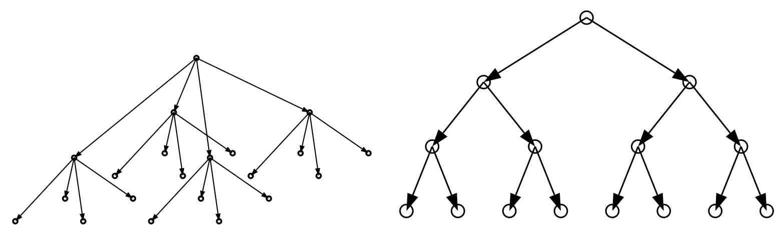


a	b	c	d	e

- (a) Original transmission phantom and (b) CBP reconstruction. Reconstructed transmission phantom using GGMRF prior with $p = 1.1$ The scale parameter σ is (c) $\hat{\sigma}_{ML} \approx \hat{\sigma}_{CBP}$, (d) $\frac{1}{2}\hat{\sigma}_{ML}$, and (e) $2\hat{\sigma}_{ML}$
- Phantom courtesy of J. Fessler, University of Michigan

Multiscale Stochastic Models

- Generate a Markov chain in scale



- Some references
  - Continuous models[2, 1, 11]
  - Discrete models[4, 11]
- Advantages:
  - Does not require a causal ordering of image pixels
  - Computational advantages of Markov chain versus MRF
  - Allows joint and marginal probabilities to be computed using forward/backward algorithm of HMM's.

References

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