Solutions EE641 Final
Fall 02

Problem 1

a) $P\{X_i = k \mid Y = y\} = P\{X_i = k \mid Y_i = y_i\}$

$$= \frac{p(y_i \mid k)\, \pi_k}{\sum_{l=0}^{M-1} p(y_i \mid l)\, \pi_l}$$

$$= \frac{e^{-\lambda_k} \lambda_k^{y_i} \pi_k}{\sum_l e^{-\lambda_l} \lambda_l^{y_i} \pi_l}$$

b) $\log p(x, y \mid \theta) =$

$$= \sum_{i=0}^{N-1} \left\{ \log p(y_i \mid x_i, \theta) + \log \pi_{x_i} \right\}$$

$$= \sum_{i=0}^{N-1} \left\{ y_i \log \lambda_{x_i} - \lambda_{x_i} - \log y_i! + \log \pi_{x_i} \right\}$$

$$= \boxed{\begin{aligned} &\sum_{i=0}^{N-1} y_i \log \lambda_{x_i} \\ &- \sum_{i=0}^{N-1} \lambda_{x_i} \\ &- \sum_{i=0}^{N-1} \log y_i! \\ &+ \sum_{i=0}^{N-1} \log \pi_{x_i} \end{aligned}} \qquad = \begin{aligned} &\sum_{K=0}^{M-1} a_K \log \lambda_K \\ &- \sum_{K=0}^{M-1} N_K \lambda_K \\ &+ S(y) \\ &+ \sum_{K=0}^{M-1} N_K \log \pi_K \end{aligned}$$

$$= \sum_{K=0}^{M-1} \left\{ a_K \log \lambda_K \quad N_K \left(\log \pi_K - \lambda_K \right) \right\} + S(y)$$

$\Rightarrow$ sufficient statistic for exponential distribution

c) $Q(\theta', \theta) =$

$$= E\left[\log p(y, X \mid \theta') \mid Y = y, \theta \right]$$

$$= E\left[\sum_{k=0}^{M-1} \left\{ a_k \log \lambda_k + N_k (\log \pi_k - \lambda_k) \right\} + s(y) \Big| Y = y, \theta \right]$$

$$= \sum_{k=0}^{M-1} \left\{ \bar{a}_k \log \lambda_k + \bar{N}_k (\log \pi_k - \lambda_k) \right\} + s(y)$$

d) $\theta'' = \arg\max_{\theta'} Q(\theta', \theta)$

$$\pi_k'' = \frac{\bar{N}_k}{N}$$

$$\lambda_k'' = \arg\max_{\lambda_k} \left\{ \bar{a}_k \log \lambda_k - \bar{N}_k \lambda_k + s(y) \right\}$$

$$= \frac{\bar{a}_k}{\bar{N}_k}$$

2)

a)

$$l(y,x) = \sum_{i\in S} -\log p(y_i|x) - \log p(x)$$

$$= \sum_{i\in S}\left\{-y_i \log(H_{i*}x) + H_{i*}X + \log y_i!\right\}$$

$$+ \sum_{\{i,j\}\in C} b_{ij}|x_i - x_j|^2 + \sum_{i\in S} a|x_i|^2$$

b) Define

$$f_i(p) = -y_i \log p + p + \log y_i!$$

Then $f_i(p)$ is a convex function of p for $\mathbb{R}\to\mathbb{R}$ for $y_i \geq 0$.

So $f_i(H_{i*}X)$ is a convex function of X $\mathbb{R}^N \to \mathbb{R}$.

The sum of convex functions is convex, so $-\log p(y|x)$ is convex.

Notice that

$$-\log p(x) = \sum_{\{i,j\}\in C} b_{ij}|x_i - x_j|^2 + \sum_{i\in S} a|x_i|^2$$

is the sum of convex terms, so it is convex.

But also notice that

$$\sum_{i \in S} a|x_i|^2 = a\|x\|^2 = a\, x^T I\, x$$

where I is a <u>strictly</u> positive <u>definit</u> matrix $\Rightarrow$ strict convexity

Therefore $l(y, x)$ is <u>strictly</u> convex.

c) Yes it exits.

Setlect any $\tilde{x} \in \mathbb{R}^N$ such that
$\forall i, \tilde{x}_i > 0$.
Then $y_i = H_{i*} \tilde{x} > 0$

$\Rightarrow f_i(H_{i*} \tilde{x}) < \infty$

$\Rightarrow \sum_{i \in S} f_i(H_{i*} \tilde{X}) < \infty$

$\Rightarrow l(y, \tilde{x}) < \infty$

Now consider the set

$$D = \{ x ; l(y, x) \leq l(y, \tilde{x}) \}$$

The D is a bounded set
(need to prove this)
$\Rightarrow$ D is compact

$\Rightarrow$ There exists a minimum

d) Yes, it is unique

Strict convexity of $l(y, x)$ in X
(and $X \geq 0$ a convex set) $\Rightarrow$ (local minimum $\Rightarrow$ global minimum)

e) Yes, it is a continuous function of y.

Strict convexity of $\ell(y, x)$ in x

$+$

Continuity of $\ell(y, x)$ in (y, x)

3)

a) $$\frac{d^2 Q(y,x)}{dy^2} > 3 \Rightarrow a(x) = \frac{3}{2}$$

$$\left.\frac{d\,Q(y,x)}{dy}\right|_{y=x} = \frac{d\,f(x)}{dx}$$

$$\left.2a(x)y + b(x)\right|_{y=x} = \frac{d\,f(x)}{dx}$$

$$2 \cdot \frac{3}{2} \cdot x + b(x) = \frac{d\,f(x)}{dx}$$

$$b(x) = \frac{d\,f(x)}{dx} - 3x$$

$$Q(x,x) = f(x)$$

$$\frac{3}{2}x^2 + \left(f'(x) - 3x\right)x + c(x) = f(x)$$

$$c(x) = f(x) - \frac{3}{2}x^2 - \left(f'(x) - 3x\right)x$$

$$= f(x) + \frac{3}{2}x^2 - f'(x)\,x$$

b)

$$f(y) \leq Q(y, x)$$

$$f(x) = Q(x, x)$$

$$f(y) - f(x) \leq Q(y, x) - Q(x, x)$$

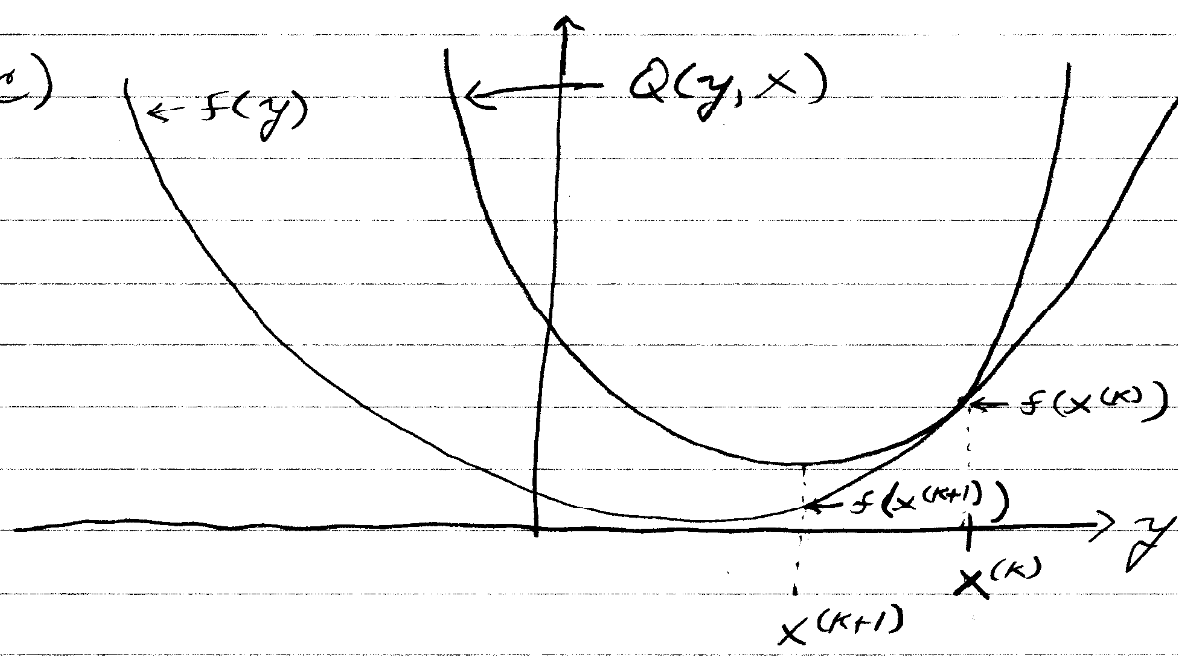
Since

$$Q(x^{(k+1)}, x^{(k)}) - Q(x^{(k)}, x^{(k)}) \leq 0$$

$$\Rightarrow \quad f(x^{(k+1)}) - f(x^{(k)}) \leq 0$$

$$\Rightarrow \quad f(x^{(k+1)}) < f(x^{(k)})$$

c)



4)

a)
$$\alpha_{n+1}(i) = \sum_{j=0}^{M-1} \alpha_n(j)\, P_{ji}\, f(Y_{n+1}|i)$$

b)
$$\beta_{n-1}(i) = \sum_{j=0}^{M-1} P_{ij}\, f(Y_n|j)\, \beta_n(j)$$

c)
$$\gamma_{n-1}(i) = \max_{j\in\{0,\dots,M-1\}} \left\{ P_{ij}\, f(Y_n|j)\, \gamma_n(j) \right\}$$