EE 641 Final Exam

Fall 2002

Name: ______________________________ Starting time: ________________________

Ending time: ________________________

Instructions

The following is a take home exam.

- Accurately fill in a start time and ending time for your exam.
- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- Answer questions precisely and completely. Credit will be subtracted for vague answers.
- Each question is worth 25 points, so make sure that you get easy parts right before doing difficult parts.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and the course text book from EE600 or an equivalent graduate course. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 463-4378 (8:00AM to 8:00PM); or office 494-0340, or send email at bouman@ecn.purdue.edu.

Good luck.

Problem 1.(25pt)

Let $\{X_i\}_{i=0}^{N-1}$ be i.i.d. random variables with $P\{X_i = k\} = \pi_k$ for $k = 0, \cdots, M-1$. Also, assume that $Y_i \in \{0, 1, 2, \cdots\}$ are conditionally independent Poisson random variables given X with

$$p(y_i|x_i) = \frac{e^{-\lambda_{x_i}} \lambda_{x_i}^{y_i}}{y_i!}$$

where $\lambda_k \geq 0$ for $0 \leq k < M$. Furthermore, define the statistic

$$S = [N_0, a_0, \cdots, N_{M-1}, a_{M-1}]$$

where

$$\begin{aligned} N_k &= \sum_{i=0}^{N-1} \delta(X_i - k) \\ a_k &= \sum_{i=0}^{N-1} Y_i \delta(X_i - k) \ . \end{aligned}$$

and define the parameter vector $\theta = (\pi, \lambda)$.

a) Write a closed form expression for $P\{X_i = k|Y = y\}$.

b) Show that $p(x, y|\theta)$ forms an exponential distribution with sufficient statistic S and parameter vector θ.

c) Write a closed form expression for $Q(\theta', \theta)$ in terms of the expectation of sufficient statistics from part b).

d) Derive the EM algorithm for estimating θ.

Problem 2.(25pt) Let X be an Gaussian MRF with pixels $X_i \geq 0$ for $i \in S$ and with Gibbs distribution

$$p(x) = \frac{1}{z} \exp \left\{ - \sum_{\{i,j\} \in \mathcal{C}} b_{i,j} |x_i - x_j|^2 - \sum_{i \in S} a |x_i|^2 \right\}$$

where $\mathcal{C}$ is the set of cliques, $a > 0$ is a positive scalar, and $b_{i,j}$ is a weighting function with the properties that

$$\begin{aligned} b_{i,j} &= b_{j,i} \geq 0 \\ 1 &= \sum_{j \in S} b_{i,j} \ , \end{aligned}$$

and let Y_i for $i \in S$ be a independent identically distributed Poisson randon variables with conditional mean

$$E\left[Y_i | X\right] = \sum_{j \in S} H_{i,j} X_j$$

where H is a linear transformation matrix with the property that for all $i \in S$, $\sum_{j \in S} H_{i,j} > 0$.

a) Compute an expression for $l(y, x) = -\log p(y, x)$.

b) Is the $l(y, x)$ not convex, convex, or strictly convex? Precisely justify your answer.

c) Does the MAP estimate exist? Precisely justify your answer.

d) Is the MAP estimate unique? Precisely justify your answer.

e) Even though the measurements Y are assumed to be discrete random variables, the form the the expression for $l(y, x)$ can be treated as a continuous function of y. Under this assumption, is the MAP estimate a continuous function of y? Precisely justify your answer.

Problem 3.(25pt)

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly convex and twice continuously differentiable function with

$$0 < \frac{d^2 f(x)}{dx^2} \leq 3$$

and let the function $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ be given by

$$Q(y, x) = a(x)y^2 + b(x)y + c(x)$$

where $a(x)$, $b(x)$, and $c(x)$ are scalar functions of x. Also consider the update equation

$$x^{(k+1)} = \arg\min_{y \in \mathbb{R}} Q(y, x^{(k)}) \ .$$

a) Find functions $a(x)$, $b(x)$, and $c(x)$ so that $Q(x, x) = f(x)$ and for all $y \neq x$

$$Q(y, x) \quad > \quad f(y)$$

b) Prove that the update equation produces a monotone decreasing sequence $f(x^{(k)})$.

c) Draw a picture illustrating the two functions, and showing why the sequence is monotone.

Problem 4.(25pt)

Let $\{X_n\}_{n=0}^N$ be a Markov chain of discrete valued random variables each taking values in the set $\{0, \cdots, M-1\}$. The distribution of X is parameterized by

$$\begin{aligned} \pi_i &= P\{X_0 = i\} \\ P_{ij} &= P\{X_{n+1} = j | X_n = i\} \ . \end{aligned}$$

Furthermore, let $\{Y_n\}_{n=1}^N$ be discrete valued random variables which are conditionally independent given X with

$$f(i|j) = P\{Y_n = i | X_n = j\} \ .$$

Define the "future" data as

$$F_n = (Y_{n+1}, \cdots, Y_N)$$

and the "past" data as

$$P_n = (Y_1, \cdots, Y_n) \ .$$

Define the two quantities

$$\begin{aligned} \alpha_n(i) &= P\{P_n, X_n = i\} \\ \beta_n(i) &= P\{F_n | X_n = i\} \\ \gamma_n(i) &= \max_{x_{n+1}, \cdots, x_N} P\{F_n | X_n = i, X_{n+1} = x_{n+1}, \cdots, X_N = x_N\} \ . \end{aligned}$$

and further define $\beta_N(i) = 1$, $\alpha_0(i) = \pi_i$, and $\gamma_N(i) = 1$.

a) Derive a recursion for $\alpha_n(i)$.

b) Derive a recursion for $\beta_n(i)$.

c) Derive a recursion for $\gamma_n(i)$.