EE641 (Fall '00)
Midterm Solution

1a) $S_x(\omega) = \frac{\sigma^2}{|1-H(\omega)|^2}$

$\frac{1}{|1-H(\omega)|^2}$ must be a stable filter

b) We know that for a GMRF

$$S_x(\omega) = \frac{\sigma_{NC}^2}{1-G(\omega)} = \frac{\sigma^2}{|1-H(\omega)|^2}$$

So $\sigma_{NC}^2 |1-H(\omega)|^2 = \sigma^2 (1-G(\omega))$

$$\sigma_{NC}^2 (\delta_s - h_s) * (\delta_s - h_{-s}) = \sigma^2 (\delta_s - g_s)$$

$$\Rightarrow \sigma_{NC}^2 \left(1 + \sum_s h_s^2\right) = \sigma^2$$

$$\sigma_{NC}^2 = \frac{\sigma^2}{1+\sum_s h_s^2}$$

$$\sigma^2 g_s = \delta_s \sigma^2 - \sigma_{NC}^2 (\delta_s - h_s) * (\delta_s - h_s) \quad (*)$$

$$g_s = \delta_s - \left(\frac{1}{1+\sum_s h_s^2}\right) (\delta_s - h_s) * (\delta_s - h_{-s})$$

c) X_s is a GMRF because g_s is an FIR filter due to the form of $(*)$.

d) $p_e(e) = \dfrac{1}{(2\pi\sigma^2)^{N/2}} \exp\left\{-\dfrac{1}{2\sigma^2}\|e\|^2\right\}$

$$p_x(x) = p_e((I-H)x)\,|I-H|$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} |I-H| \exp\left\{-\frac{1}{2\sigma^2}\|(I-H)x\|^2\right\}$$

$$= \frac{1}{(2\pi\sigma^2)^{N/2}} |I-H| \exp\left\{-\frac{1}{2\sigma^2} x^* B x\right\}$$

$$B \triangleq (I-H)^*(I-H)$$

2)

a) $Z = (2\pi)^{\frac{N}{2}} |B|^{-1/2}$

b) $\log p(x) = -\frac{1}{2} x^T B x + \text{const}$

$= \log p(x_s | x_r\ r \neq s) + \log p(x_r\ r \neq s) = -\frac{1}{2} \frac{1}{\sigma_{NC}^2} (x_s - \mu)^2 + \text{const}$

$$\nabla \left(-\frac{1}{2} x^T B x\right) = 2\left(-\frac{1}{2}\right) x^T B$$

$$\frac{\partial}{\partial x_s}\left(-\frac{1}{2} x^T B x\right) = \left[\nabla\left(-\frac{1}{2} x^T B x\right)\right]_s$$

$$= -x^T B_{*s}$$

$$= -\sum_r x_r B_{rs}$$

$$\frac{\partial^2}{\partial x_s^2}\left(-\frac{1}{2} x^T B x\right) = \frac{\partial}{\partial x_s}\left(-\sum_r x_r B_{rs}\right)$$

$$= -B_{ss}$$

$$\frac{\partial}{\partial x_s}\left(-\frac{1}{2}\frac{1}{\sigma_{NC}^2}(x_s-\mu)^2\right) = -\frac{1}{\sigma_{NC}^2}(x_s-\mu)$$

$$\frac{\partial^2}{\partial x_s^2}\left(-\frac{1}{2}\frac{1}{\sigma_{NC}^2}(x_s-\mu)^2\right) = -\frac{1}{\sigma_{NC}^2}$$

So

$$-\frac{1}{\sigma_{vc}^2} = -B_{ss}$$

$$\boxed{\sigma_{vc}^2 = (B_{ss})^{-1}}$$

$$-\sum_r x_r B_{rs} = -\frac{1}{\sigma_{vc}^2}(x_s - \mu)$$

$$-x_s B_{ss} - \sum_{r \neq s} x_r B_{rs} = -B_{ss}(x_s - \mu)$$

$$\sum_{r \neq s} x_r \frac{B_{rs}}{B_{ss}} = -\mu$$

$$\boxed{\mu = \sum_{r \neq s} x_r \left(\frac{-B_{rs}}{B_{ss}}\right)}$$

$$E[x_s \mid x_i \;\; i \neq s] = \sum_{i \neq s} x_i \left(-\frac{B_{is}}{B_{ss}}\right)$$

c) From homework

$$p(x/y) \sim N(\mu, R_{x|y})$$

where $\mu = R_x (R_x + R_n)^{-1} y$

and $R_{x|y} = (R_n^{-1} + R_x^{-1})^{-1}$

For this problem $R_x = B^{-1}$ and $R_n = I\sigma^{+2}$

(continued)

$$\mu = B^{-1}(\sigma_n^2 I + B^{-1})^{-1} y = (I + \sigma_n^2 B)^{-1} y$$

$$R_{x|y} = \left(\frac{1}{\sigma_n^2} I + B\right)^{-1}$$

$$p(x|y) =$$

$$\frac{1}{(2\pi)^{N/2}} \left(\frac{1}{\sigma_n^2} I + B\right)^{1/2} \exp\left\{-\frac{1}{2}(x-\mu)^T\left(\frac{1}{\sigma_n^2} I + B\right)(x-\mu)\right\}$$

d) $E[x|y] = R_x(R_x + R_n)^{-1} y$

$$\hat{x} = B^{-1}(\sigma_n^2 I + B^{-1})^{-1} y$$

$$\hat{x} = (I + \sigma_n^2 B)^{-1} y$$

$$= \frac{1}{\sigma_n^2}\left(\frac{1}{\sigma_n^2} I + B\right)^{-1} y$$

e) Since B is a circulant matrix, Bz is circulan convolution of some filter h_s with the signal z. The filter h_s is given by the equation

$$h_{(i-j) \bmod N} = B_{ij}$$

Next define $\Lambda = \mathrm{diag}(H(0), \ldots, H(N-1))$

where $H(k) = DFT\{h_n\}$

and let T be the DFT transform matrix. Then we know that

$$B = T^{-1} \Lambda T$$

So

$$\hat{x} = B^{-1}(\sigma_n^2 I + B^{-1})^{-1} y$$

$$= (T^{-1}\Lambda T)^{-1}(\sigma_n^2 I + (T^{-1}\Lambda T)^{-1})^{-1} y$$

$$= (T^{-1}\Lambda^{-1} T)(\sigma_n^2 I + T^{-1}\Lambda^{-1} T)^{-1} y$$

$$= (T^{-1}\Lambda^{-1} T)(T^{-1}(\sigma_n^2 I + \Lambda^{-1}) T)^{-1} y$$

$$= (T^{-1}\Lambda^{-1} T) T^{-1}(\sigma_n^2 I + \Lambda^{-1})^{-1} T y$$

$$= T^{-1} \Lambda^{-1}(\sigma_n^2 I + \Lambda^{-1})^{-1} T y$$

$$= T^{-1}(\sigma_n \Lambda + I)^{-1} T y$$

Summary

Compute DFT of $\cancel{y}$.

$$Y(k) = \sum_{n=0}^{N-1} y_n \, e^{-2\pi j \frac{nk}{N}}$$

$$Z(k) = \frac{Y(k)}{1 + \sigma_n^2 H(k)}$$

$$\hat{x}_n = \frac{1}{N} \sum_{k=0}^{N-1} Z(k) \, e^{j 2\pi \frac{nk}{N}}$$

3.a) We need λ and e such that (use indexing from 0 to $N-1$)

$$\sum_{j=0}^{N-1} H_{ij}\, e_j = e_i \lambda$$

$$= \underbrace{\sum_{j=0}^{N-1} X_{(i-j) \bmod N}\, e_j}_{\text{circular convolution}} = e_i \lambda$$

Choose $e_n = e^{j2\pi \frac{nk_0}{N}}$, Then

$$DFT\left\{ e^{2\pi j \frac{nk_0}{N}} \right\} = N\delta(k-k_0) \qquad 0 \le k \le N-1$$

So

$$\sum_{j=0}^{N-1} X_{(i-j) \bmod N}\, e_j = e_i\, X(k_0)$$

The N eigenvalue/eigenvector pairs are

$$e_i^{(k)} = \exp\left\{ 2\pi j \frac{ik}{N} \right\}$$

$$\lambda^{(k)} = X(k)$$

b) We know that

$$|H| = \prod_{k=0}^{N-1} \lambda^{(k)}$$

$$= \prod_{k=0}^{N-1} X(k) = \prod_{k=0}^{N-1} |X(k)|$$

c) $$\log|H| = \sum_{k=0}^{N-1} \log X(k)$$

$$X(k) = X(e^{j\omega})\Big|_{\omega = 2\pi \frac{k}{N}}$$

where
$$X(e^{j\omega}) = \underbrace{\sum_{n=0}^{N-1} x_n (e^{j\omega})^{-n}}_{DTFT}$$

$$\lim_{N\to\infty} \frac{1}{N} \log|H| = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} \log|X(k)|$$

$$= \int_0^1 \log|X(e^{j2\pi f})|\, df$$

$\omega = 2\pi f$
$d\omega = 2\pi\, df$
$$= \frac{1}{2\pi} \int_0^{2\pi} \log|X(e^{j\omega})|\, d\omega$$

4 a) $$p(y \mid x, \theta) = \prod_{n=0}^{N-1} p(y_n \mid x_n, \theta)$$

$$= \prod_{n=0}^{N-1} \left\{ \frac{e^{-\lambda_{x_n}}}{y_n!} \lambda_{x_n}^{y_n} \right\}$$

$$p(y, x \mid \theta) = \prod_{n=0}^{N-1} \left\{ \frac{e^{-\lambda_{x_n}}}{y_n!} \lambda_{x_n}^{y_n} \pi_{x_n} \right\}$$

b) $$f(x_n \mid y_n, \theta) = \frac{p(y_n, x_n \mid \theta)}{\sum_{x_n=0}^{M-1} p(y_n, x_n \mid \theta)}$$

$$= \frac{\cancel{\frac{1}{y_n!}} e^{-\lambda_{x_n}} \lambda_{x_n}^{y_n} \pi_{x_n}}{\sum_{K=0}^{M-1} \cancel{\frac{1}{y_n!}} e^{-\lambda_K} \lambda_K^{y_n} \pi_K}$$

$$= \frac{e^{-\lambda_{x_n}} \lambda_{x_n}^{y_n} \pi_{x_n}}{\sum_{K=0}^{M-1} e^{-\lambda_K} \lambda_K^{y_n} \pi_K}$$

$$= \frac{1}{\sum_{K=0}^{M-1} e^{\lambda_{x_n} - \lambda_K} \left(\frac{\lambda_n}{\lambda_{x_n}} \right)^{y_n} \frac{\pi_K}{\pi_{x_n}}}$$

c) $N_k = \sum_{n=0}^{N-1} \delta(x_n - k)$

$$\hat{\pi}_k = \frac{N_k}{N}$$

$$\hat{\lambda}_k = \frac{1}{N_k} \sum_{n=0}^{N-1} y_n \, \delta(x_n - k)$$

d) $Q(\theta';\theta) = E[\log p(y, x|\theta') \mid Y, \theta]$

$$= E\left[\sum_{n=0}^{N-1} \{ y_n \log \lambda_{x_n} - \lambda_{x_n} - \log y_n + \log \pi_{x_n} \} \middle| Y, \theta \right]$$

$$= \sum_{n=0}^{N-1} \sum_{x_n=0}^{M-1} \{ y_n \log \lambda_{x_n} - \lambda_{x_n} - \log y_n + \log \pi_{x_n} \} f(x_n | y_n, \theta)$$

$$\hat{\lambda}_k = \arg\max_{\lambda_k} \sum_{n=0}^{N-1} \{ y_n \log \lambda_k - \lambda_k \} f(k | y_n, \theta)$$

$$\hat{\pi} = \arg\max_{\sum_{n=0}^{M-1} \pi_n = 1} \sum_{n=0}^{N-1} \sum_{k=0}^{M-1} (\log \pi_k) f(k | y_n, \theta)$$

<u>Solution</u>

$$\hat{N}_k = \sum_{n=0}^{N-1} f(k | y_n, \theta)$$

$$\hat{\lambda}_k = \frac{1}{N_k} \sum_{n=0}^{N-1} y_n f(k | y_n, \theta)$$

$$\hat{\pi}_k = \frac{N_k}{N}$$

5) a) $$N_{l,m} = \sum_{n=1}^{2N} \delta(x_n - m)\,\delta(x_{n-1} - l)$$

$$\hat{P}_{l,m} = \frac{N_{l,m}}{\sum_{k=0}^{M-1} N_{l,k}}$$

b) $$p(x_{n+1}, x_n \mid x_{n-1}) = P_{x_{n-1} x_n} P_{x_n x_{n+1}}$$

$$p(x_n \mid x_{n+1}, x_{n-1}) = \frac{p(x_{n+1}, x_n \mid x_{n-1})}{\sum_{x_n=0}^{M-1} p(x_{n+1}, x_n \mid x_{n-1})}$$

$$= \frac{P_{x_{n-1} x_n} P_{x_n x_{n+1}}}{\sum_{k=0}^{M-1} P_{x_{n-1} k} P_{k x_{n+1}}}$$

$$\triangleq f(x_n \mid x_{n+1}, x_{n-1})$$

c) $Q(\theta', \theta) = E\left[\log p(x | \theta') \,/\, x_{even}, \theta \right]$

$$p(x|\theta) = \prod_{l=0}^{M-1} \prod_{m=0}^{M-1} P_{lm}^{N_{lm}}$$

$$\log p(x|\theta) = \sum_{l=0}^{M-1} \sum_{m=0}^{M-1} N_{lm} \log P_{lm}$$

$$Q(\theta', \theta) = \sum_{l=0}^{M-1} \sum_{m=0}^{M-1} \hat{N}_{l,m} \log P_{lm}$$

where

$$\hat{N}_{l,m} = E\left[N_{l,m} \,/\, x_{even}, \theta \right]$$

$$N_{l,m} = \sum_{n=1}^{N} \left\{ \delta(x_{2n} - m)\, \delta(x_{2n-1} - l) + \delta(x_{2n-1} - m)\, \delta(x_{2n-2} - l) \right\}$$

So

E-step

$$\hat{N}_{l,m} = \sum_{n=1}^{N} \left\{ f(l | x_{2n}, x_{2n-2})\, \delta(x_{2n} - m) + f(m | x_{2n}, x_{2n-2})\, \delta(x_{2n-2} - l) \right\}.$$

$$\hat{\theta} = \arg\max_{\theta} \sum_{l,m} \hat{N}_{lm} \log P_{lm}$$

M-step

$$\hat{P}_{lm} = \frac{\hat{N}_{lm}}{\sum_{k=0}^{M-1} \hat{N}_{l,k}}$$