EE 641
Final Exam Solution
12/1/00

1) a) • $u(x)$ must have a minimum on $\Omega$ because it is a continuous function on a compact set.

• The minimum of $u(x)$ must be unique because $u(x)$ is a strictly convex function and $\Omega$ is a convex set.

b) For each iteration $k$, we know that

$$u(\hat{x}^{(k+1)}) \le u(\hat{x}^{(k)})$$

Therefore $u(\hat{x}^{(k)})$ is a monotone decreasing sequence. Since $u(x)$ is also bounded below,

$$\lim_{k \to \infty} u(\hat{x}^{(k)}) \text{ must exist.}$$

YES

c) $p(x_s | x_r \; r \neq s) = \frac{1}{z} \exp\left\{ -V(x_s | x_r \; r \in \partial s) \right\}$

where

$$V(x_s | x_r \; r \in \partial s) =$$
$$(y_s - x_s)^2 + \sum_{r \in \partial s} \left\{ \log(x_s - x_r + 2) + \log(x_r - x_s + 2) \right\}$$

$$z = \int_0^1 \exp\left\{ -V(x_s | x_r \; r \in \partial s) \right\} dx_s$$

2) a) This is the Metropolis algorithm. However, the sampling distribution is

$$q(z_s \mid x_s) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z_s - x_s)^2} & \text{for } z_s \in [0,1] \\ a & \text{for } z_s = 0 \\ b & \text{for } z_s = 1 \end{cases}$$

where

$$a = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x_s)^2} dz$$

$$b = \int_{1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-x_s)^2} dz$$

However, $q(z_s \mid x_s)$ is NOT a symmetric density function because

$$q(z_s \mid x_s) \neq q(x_s \mid z_s)$$

b) Replace ~~the~~ step 1(a) v. with the step

$$z_s \leftarrow z_s \bmod 1$$

then

$$q(z_s \mid x_s) = q(x_s \mid z_s)$$

Alternatively, use the Hastings-Metropolis algorithm to modify step 1(a) vii.

3 a)

$$p(y|x) = \frac{1}{(2\pi)^{N/2}} \exp\left\{ -\frac{1}{2} (Y - Ax)^t (Y - Ax) \right\}$$

$$p(x) = \frac{1}{z} \exp\left\{ - \sum_{\{s,r\} \in C} \rho( (x_s - x_r)/\sigma ) \right\}$$

$$\hat{X}_{MAP} = \arg\min_{x \in \mathbb{R}^N} \underbrace{\left\{ \frac{1}{2} \|Y - AX\|^2 + \sum_{\{s,r\} \in C} \rho((x_s - x_r)/\sigma) \right\}}_{C(x)}$$

b) Let $\quad C^* = \inf_{x \in \mathbb{R}^N} C(x)$ and $C_0 = C(0)$

~~Let $\lambda_0$~~

Let $\lambda_0 > 0$ be the smallest eigenvalue of $A^t A$. Define

$$S = \left\{ X : \|X - A^{-1} Y\|^2 \leq 2 \frac{C_0 + 1}{\lambda_0} \right\}$$

Then $\forall x \in S$, we have that

$$C(x) \geq \frac{1}{2} \| Y - A^{-1} X \|^2_{A^t A}$$

$$\geq \frac{1}{2} \| Y - A^{-1} X \|^2 \lambda_0$$

$$\geq (C_0 + 1)$$

Therefore,

$$\inf_{x \in S} C(x) > C_0 \geq \inf_{x \in S} C(x)$$

$$\Rightarrow \quad \inf_{x \in S} C(x) = \inf_{x \in \mathbb{R}^N} C(x)$$

Since $S$ is a compact set, there exists $X^*$ such that

$$C(X^*) = \inf_{x \in S} C(x)$$

$$= \inf_{x \in \mathbb{R}^N} C(x)$$

c) Since $C(x)$ is a strictly convex function, any local minimum must also be a ~~global~~ unique global minimum. Clearly, $X^*$ is a local minimum (since it is a global minumum.). Therefore, it must be a unique global minumum.

d) Notice that $C(x,y) = \frac{1}{2}\|Y - Ax\|^2 + \sum_{\{s,r\} \in C} \rho((x_s - x_r)/\sigma)$

is a continuous function of $(x,y)$ and a strictly convex function of $y$

$\Rightarrow$ $\hat{X}_{MAP}$ is a continuous function of $Y$.

e) $\nabla_x \frac{1}{2}\|Y - Ax\|^2 = (Y - Ax)^t(-A)$

$$= -(Y - Ax)^t A$$

$$\frac{d}{dx_s} \log p(x) = \sum_{r \in \partial s} \frac{1}{\sigma} \rho'\left(\frac{x_s - x_r}{\sigma}\right)$$

$$\left[\nabla_x c(x)\right]_s = -(Y - AX)^t A_{*s} + \sum_{r \in \partial s} \frac{1}{\sigma} \rho'\left(\frac{x_s - x_r}{\sigma}\right)$$

$$X \leftarrow X - \mu\left[\nabla c(x)\right]$$

$$= X - \mu\left[-(Y - AX)^t A_{*s} + \sum_{r \in \partial s} \frac{1}{\sigma} \rho'\left(\frac{x_s - x_r}{\sigma}\right)\right]$$

$$= X + \mu\left[(Y - AX)^t A_{*s} - \sum_{r \in \partial s} \frac{1}{\sigma} \rho'\left(\frac{x_s - x_r}{\sigma}\right)\right]$$

4) a) $\alpha_0(i) = \pi_i$

b) $$\alpha_{n+1}(i) = \sum_{j=0}^{M-1} f(Y_{n+1}/i)\, P_{ji}\, \alpha_n(j)$$

c) $$\beta_n(i) = \sum_{j=0}^{M-1} f(Y_{n+1}/j)\, \beta_{n+1}(j)\, P_{ij}$$

d) $$p(Y, x_n) = \cancel{\alpha_n(i)\, \beta_n(j)}\ \alpha_n(x_n)\, \beta_n(x_n)$$

$$p(x_n/Y) = \frac{\alpha_n(x_n)\, \beta_n(x_n)}{\sum_{i=0}^{M-1} \alpha_n(i)\, \beta_n(i)}$$

~~e) Define $l_n(i)$ = log $\alpha_{n+1}(i)$~~