EE 641 Midterm Exam

Fall 2000

Name: ______________________ Starting time: ______________________

Ending time: ______________________

Instructions

The following is a take home exam.

- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- Answer questions precisely and completely. Credit will be subtracted off for vague answers.
- Each question is worth 20 points, so make sure that you get easy parts right before doing difficult parts.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and course text books from graduate courses taught in ECE. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 463-4378 (8:00AM to 8:00PM); or office 494-0340, or send email at bouman@ecn.purdue.edu.

Good luck.

Problem 1.(20pt)

Let X_s be an $N \times N$ 2-D zero mean Gaussian random field. Let h_s be a 2-D symmetric FIR filter (i.e. $h_s = h_{-s}$) with $h_{(0,0)} = 0$. Also, let

$$E_s = (\delta_s - h_s) * X_s$$

where E_s are i.i.d. random variables distributed as $N(0, \sigma^2)$.

a) Compute the power spectrum for X_s. Clearly state any assumptions that you need to make.

b) Compute the MMSE noncausal prediction filter g_s and its associated prediction variance σ_{nc}^2.

c) Is X_s a GMRF? Justify your answer.

Using matrix notation, we may write that

$$E = (I - H)X$$

where H is a Toeplitz-block-Toeplitz matrix that represents the filter h_s.

d) Use matrix notation to write the density function for X.

Problem 2.(20pt)

Let X be an N dimensional 1-D randon vector with density function

$$p(x) = \frac{1}{z} \exp\left\{ -\frac{1}{2} x^t B x \right\}$$

where B is an $N \times N$ symmetric circulant positive definite matrix.

For each of the following questions, justify your answer.

a) Compute z.

b) Compute $E[X_s | X_i \; i \neq s]$.

Let $Y = X + N$ where N is a vector with distribution $N(0, I\sigma^2)$.

c) Compute $p(x|y)$.

d) Compute $E[x|y]$.

e) Use the DFT to efficiently compute $E[x|y]$.

Problem 3.(20pt)
Let x_n be a 1-D discrete time function, and let H be an $N \times N$ circulant matrix with

$$H_{i,j} = x_{(i-j) mod N} \ .$$

Furthermore, let

$$X(k) = \sum_{n=0}^{N-1} x_n e^{-j2\pi \frac{nk}{N}}$$

a) Find N complex eigenvalues and eigenvectors for the matrix H.

b) Compute the determinant for the matrix H.

c) Find a closed form expression for

$$\lim_{N \to \infty} \frac{\log |H|}{N}$$

Problem 4.(20pt)
Let $\{X_n\}_{n=0}^{N-1}$ be i.i.d. random variables with

$$P\{X_n = k\} = \pi_k$$

for $0 \leq k \leq M-1$.

Let $\{Y_n\}_{n=0}^{N-1}$ be conditionally independent random variables given X, where the conditional distribution of Y_n given X_n is Poisson with mean λ_{X_n}. Further define $\theta = (\pi, \lambda)$.

a) Derive $p(y, x|\theta)$.

b) Derive $f(x_n|y_n, \theta) = p(x_n|y_n, \theta)$.

c) Derive the ML estimate of θ given both Y and X.

d) Derive the EM update equations for estimation of θ from Y.

Problem 5.(20pt)

Let $\{X_n\}_{n=0}^{2N}$ be a Markov Chain with

$$P\{X_0 = k\} = \frac{1}{M}$$

for $0 \le k \le M-1$, and

$$P\{X_n = m | X_{n-1} = l\} = P_{l,m}$$

for $0 \le l, m \le M-1$.

a) Derive the ML estimate of $P_{l,m}$ given $\{X_n\}_{n=0}^{2N}$.

b) Derive an expression for

$$f(x_n | x_{n-1}, x_{n+1}) \triangleq P\{X_n = x_n | X_{n-1} = x_{n-1}, X_{n+1} = x_{n+1}\}$$

c) Use the EM algorithm to derive an iterative procedure for computing the ML estimate of $P_{l,m}$ from $\{X_{2n}\}_{n=0}^{N}$.