EE 641 Final Exam

Fall 2000

Name: ______________________ Starting time: ______________________

Ending time: ______________________

Instructions

The following is a take home exam.

- You are allowed 24 hours to complete the exam. Hand in the exam after that period **whether or not you have completed it.**
- Answer questions precisely and completely. Credit will be subtracted off for vague answers.
- Each question is worth 25 points, so make sure that you get easy parts right before doing difficult parts.
- You should not discuss these problems with any other person. In addition, you should not communicate with any other student in the class during the test period. Your are allowed to use all class notes and handouts, and course text books from graduate courses taught in ECE. Your are not allowed to use supplementary information from the library, or publications not handed out in class.
- If you have any questions, call me at home 463-4378 (8:00AM to 8:00PM); or office 494-0340, or send email at bouman@ecn.purdue.edu.

Good luck.

Problem 1.(25pt)

Let $X_s \in [0,1]$ be an image where s is a position in a finite lattice S. Futhermore, let ∂s define a neighborhood system on S with C being the set of all cliques. Consider the functional

$$u(x) = \sum_{s\in S}(y_s - x_s)^2 - \sum_{\{s,r\}\in C} \{\log(X_s - X_r + 2) + \log(X_r - X_s + 2)\}$$

where for each s, $y_s \in \mathbb{R}$.

a) What do you know about the solution to the problem

$$\hat{X} = \arg\min_{x\in\Omega} u(x) \tag{1}$$

where $\Omega = [0,1]^N$.

b) Let $\hat{X}^{(k)}$ be the k^{th} update of a coordinate decent algorithm. Does the sequency $u(\hat{X}^{(k)})$ converge? Prove your answer.

c) Assume that X is distributed as

$$\frac{1}{z}\exp\{-u(x)\} \ .$$

Compute the conditional distribution

$$p(x_s | x_r \ r \neq s)$$

Problem 2.(25pt)

Let $X_s \in [0,1]$ be an image where s is a position in the lattice S. Assume that X is distributed as

$$p(x) = \frac{1}{z} \exp\{-u(x)\}$$

for $X \in \Omega$ where $\Omega = [0,1]^N$, where $u(x)$ is the energy functional of a Gibbs distribution. Your friend writes the following iterative program to sample from the distribution $p(x)$.

1. For each iteration
   (a) For each pixel $s \in S$
      i. Generate a new Gaussian random number W with mean 0 and variance 1.
      ii. Generate a new uniformly distributed random number T on $[0,1]$.
      iii. Set $Z \leftarrow X$
      iv. Set $Z_s \leftarrow X_s + W$
      v. Set $Z_s \leftarrow \min\{1, \max\{0, Z_s\}\}$
      vi. Set $\Delta u = u(Z) - u(X)$
      vii. If $T < e^{-\Delta u}$, set $X \leftarrow Z$

a) Find the flaw in your friends algorithm.

b) Propose an alternative algorithm which will converge to the desired distribution.

Problem 3.(25pt)

Let $X \in \mathbb{R}^N$ be an image, and let X_s be a pixel where s is a position in the lattice S. Let ∂s define a 4-point neighborhood system on S with C being the set of all cliques. Furthermore, assume that X is distributed as

$$p(x) = \frac{1}{z} \exp\left\{ - \sum_{\{s,r\} \in C} \rho((X_s - X_r)/\sigma) \right\}$$

where $\rho(\Delta)$ is a symetric, convex and continuously differentiable function with $\rho(0) = 0$. Also assume that Y is measured data formed by

$$Y = AX + W$$

where W is Gaussian noise with distribution $N(0, I)$, and A is a non-singular square matrix. (Here, I is the identity matrix.)

a) Derive an expression for the MAP estimate of X given Y.

b) Prove that the MAP estimate exists.

c) Prove that the MAP estimate is unique.

d) Use a theorem stated in class to prove that the MAP estimate is a continuous function of the data?

e) Compute expressions for gradient decent optimization.

Problem 4.(25pt)

Let $\{X_n\}_{n=0}^N$ be a Markov chain of discrete valued random variables each taking values in the set $\{0, \cdots, M-1\}$. The distribution of X is parameterized by

$$\begin{aligned}\pi_i &= P\{X_0 = i\} \\ P_{ij} &= P\{X_{n+1} = j | X_n = i\} \ .\end{aligned}$$

Furthermore, let $\{Y_n\}_{n=1}^N$ be discrete valued random variables which are conditionally independent given X with

$$f(i|j) = P\{Y_n = i | X_n = j\} \ .$$

Define the "future" data as

$$F_n = (Y_{n+1}, \cdots, Y_N)$$

and the "past" data as

$$P_n = (Y_1, \cdots, Y_n) \ .$$

Define the two quantities

$$\begin{aligned}\alpha_n(i) &= P\{P_n, X_n = i\} \\ \beta_n(i) &= P\{F_n | X_n = i\} \ .\end{aligned}$$

And further define $\beta_N(i) = 1$.

a) What is the value of $\alpha_0(i)$?

b) Compute an expression for $\alpha_{n+1}(i)$ in terms of $\alpha_n(i)$.

c) Compute an expression for $\beta_n(i)$ in terms of $\beta_{n+1}(i)$.

d) Use your results to compute an expression for $P\{X_n = i | Y\}$.